Image reconstruction using Compressed Sensing

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Outline

1. Formulating Compressed Sensing
2. Applications
3. Conclusions
Deconvolution

**CLEAN**
- Local iterative deconvolution
- Matching Pursuit
- Implicitly implies sparsity

**MEM**
- Global minimization problem
- Assumes an entropic prior

Both methods are flexible enough to consider a variety of bases (Dirac, wavelet etc)
The main premise of CS is that although our signal is not necessarily sparse in real space or Fourier space, it is sparse or compressible in some basis.

If we consider a real signal $x = \{x_i\}_{1 \leq i \leq N}$ and define a real basis $\Psi = \{\psi_{i\omega}\}_{1 \leq i \leq N; 1 \leq \omega \leq T}$, then we can say that the decomposition $\alpha = \{\alpha_{\omega}\}_{1 \leq \omega \leq T}$.

$x = \Psi \alpha$

...is spare or compressible if it contains only $K << N$ non-zero or significant co-efficients.
If we then probe the signal using $m$ real linear measurements (visibilities) $y = \{y_r\}_{1 \leq r \leq m}$ in some sensing basis $\Phi = \{\phi_{ri}\}_{1 \leq r \leq m; 1 \leq i \leq N}$ and these measurements are possibly affected by some independent and identically distributed noise: $n = \{n_r\}_{1 \leq r \leq m}$, so that:

$$y = \Theta \alpha + n,$$

where $\Theta = \Phi \Psi \in \mathbb{R}^{m \times T}$.
Restrictive Isometry

- Defining the $\ell_p$ norm, $\|u\|_p = \left( \sum_{i=1}^{Q} |u_i|^p \right)^{1/p}$.
- By definition the matrix $\Theta$ satisfies a RIP of order $K$ if there exists a constant $\delta_K < 1$ such that

$$
(1 - \delta_K)\|\alpha_K\|_2^2 \leq \|\Theta \alpha_K\|_2^2 \leq (1 + \delta_K)\|\alpha_K\|_2^2
$$

The RIP
Satisfying the RIP

- The incoherence of the sensing matrix \( \Phi \) with the sparsity basis \( \Psi \) will satisfy the RIP if the number of measurements \( (m) \) is large enough relative to the sparsity \( K \).
- For radio interferometry the RIP is satisfied if

\[
K \leq \frac{C m}{\mu^2 \ln^4 N}.
\]

- \( \mu \) is the mutual coherence of the elements of the Fourier basis and the elements of the sparsity basis:

\[
\mu = \sqrt{N} \max \mid \langle \phi_e | \psi_{e'} \rangle \mid.
\]
Basis Pursuit

- the $\ell_1$ norm of the vector $\alpha$ is simply the sum of the absolute values of the vector components:
  $$||\alpha||_1 = \sum_{\omega=1}^{T} |\alpha_\omega|.$$  

→ Laplacian Prior

The Optimization Problem

$$\min ||\alpha'||_1 \text{ s.t. } ||y - \Theta \alpha'||_2 \leq \epsilon$$

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If the solution of the BP is $\alpha^*$ then the corresponding recovered signal is

$$x^* = \Psi \alpha^*.$$  

**Solutions**

- CS shows that if $\Theta$ satisfies a RIP of order $2K$ with $\delta_{2K} < \sqrt{2} - 1$ then the solution $x^*$ provides an accurate reconstruction.
- It can be said to be *optimal* in the sense that exactly sparse signals in the absence of noise are recovered exactly.
- In the presence of noise very strong stability results are obtained.
We consider 5 different sets of coverage in $uv$ with different % coverage of the Fourier plane.

Two examples:
1. a field filled with multi-variate compact sources
2. a CMB cosmic string signal simulation

$\text{SNR}^{(s,s')} = -20 \log_{10} \frac{\sigma(s-s')}{\sigma(s)}$
Cosmic Strings

- Topological defects in the CMB.
- $Λ$CDM cosmology.
- String signal is well modelled by GGDs in wavelet space

**GGDs**

$$\pi_j(\alpha_\omega) \propto \exp\left[-\left[\frac{\alpha_\omega}{\rho u_j}\right] v_j\right]$$

**s-norm**

$$\pi(\alpha) \propto \exp -\|\alpha\|_s$$

$$\|\alpha\|_s = \sum_\omega \left|\frac{\alpha_\omega}{\rho u_j}\right| v_j$$
The three methods

**BP**

\[
\min \|\alpha'\|_1 \text{ s.t. } \|y - \Theta \alpha'\|_2 \leq \epsilon
\]

**BP⁺**

\[
\min \|\bar{x}'\|_1 \text{ s.t. } y = \bar{\Phi}_r \bar{x}' \text{ and } \bar{x}' \geq 0.
\]

**SBP**

\[
\min \|\alpha'\|_s \text{ s.t. } \|\bar{y} - W_{cmb} \Phi_r \Psi_s \alpha'\|_2 \leq \epsilon
\]
Compact Sources

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Cosmic Strings

Image reconstruction using Compressed Sensing
Image reconstruction using Compressed Sensing
A new framework for image reconstruction in interferometry
Simple BP provides the same image fidelity as CLEAN
BP is more rapid than CLEAN in terms of no. iterations and computation time
Prior statistical knowledge of the signal can greatly improve the reconstruction