Image reconstruction using Compressed Sensing

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Outline

Deconvolution

CLEAN

- Local iterative deconvolution
- Matching Pursuit
- Implicitly implies sparsity

MEM

- Global minimization problem
- Assumes an entropic prior

Both methods are flexible enough to consider a variety of bases (Dirac, wavelet etc)

Sparsity

- The main premise of CS is that although our signal is not necessarily sparse in real space or Fourier space, it is sparse or compressible in *some basis*.
- If we consider a real signal $x = \{x_i\}_{1 \leq i \leq N}$
- and define a real basis $\Psi = \{\Psi_{i\omega}\}_{1 \leq i \leq N; 1 \leq \omega \leq T}$
- Then we can say that the decomposition $\alpha = \{\alpha_\omega\}_{1 \leq \omega \leq T}$.
$$x = \Psi \alpha$$
- ...is spare or compressible if it contains only $K << N$ non-zero or significant co-efficients.

- If we then probe the signal using m real linear measurements (visibilities) $y = \{y_r\}_{1 \leq r \leq m}$ in some sensing basis $\Phi = \{\phi_{ri}\}_{1 \leq r \leq m; 1 \leq i \leq N}$
- and these measurements are possibly affected by some independent and identically distributed noise: $n = \{n_r\}_{1 \leq r \leq m}$, so that:

$$y = \Theta\,\alpha + n, \text{ where } \Theta = \Phi\,\Psi \in \mathbb{R}^{m \times T}$$

Restricted Isometry

- Defining the ℓ_p norm, $||u||_p = \left(\sum_{l=1}^{Q} |u_l|^p\right)^{1/p}$.
- By definition the matrix Θ satisfies a RIP of order K if there exists a constant $\delta_K < 1$ such that

The RIP

$$(1-\delta_K)||\alpha_K||_2^2 \leq ||\Theta\,\alpha_K||_2^2 \leq (1+\delta_K)||\alpha_K||_2^2$$

Satisfying the RIP

- The incoherence of the sensing matrix Φ with the sparsity basis Ψ will satisfy the RIP if the number of measurements (m) is large enough relative to the sparsity K.
- For radio interferometry the RIP is satisfied if

$$K \leq \frac{C\,m}{\mu^2 \ln^4 N}.$$

- μ is the mutual coherence of the elements of the Fourier basis and the elements of the sparsity basis:

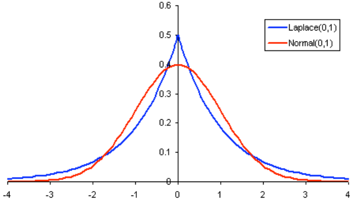
$$\mu = \sqrt{N}\,\max\,|\langle\phi_e|\psi_{e'}\rangle|.$$

Basis Pursuit

- the ℓ_1 norm of the vector α is simply the sum of the absolute values of the vector components:

$$||\alpha||_1 = \sum_{\omega=1}^{T} |\alpha_\omega|.$$

$\longrightarrow$ Laplacian Prior



The Optimization Problem

$$\min ||\alpha'||_1 \text{ s.t. } ||y - \Theta\, \alpha'||_2 \leq \epsilon$$

Recovery

If the solution of the BP is α^* then the corresponding recovered signal is

$$x^* = \Psi\,\alpha^*.$$

Solutions

- CS shows that if Θ satisfies a RIP of order $2K$ with $\delta_{2K} < \sqrt{2} - 1$ then the solution x^* provides an accurate reconstruction.
- It can be said to be *optimal* in the sense that exactly sparse signals in the absense of noise are *recovered exactly*.
- In the presence of noise very strong stability results are obtained.

Interferometer Data

- We consider 5 different sets of coverage in *uv* with different % coverage of the Fourier plane
- Two examples:
  1. a field filled with multi-variate compact sources
  2. a CMB cosmic string signal simulation
- $\mathrm{SNR}^{(s,s')} = -20\log_{10}\frac{\sigma^{(s-s')}}{\sigma^{(s)}}$

Cosmic Strings

- Topological defects in the CMB.
- ΛCDM cosmology.
- String signal is well modelled by GGDs in wavelet space

GGDs

$$\pi_j(\alpha_\omega) \propto \exp\left[-\left[\frac{\alpha_\omega}{\rho u_j}\right]^{v_j}\right]$$

s-norm

$$\pi(\alpha) \propto \exp -||\alpha||_s$$
$$||\alpha||_s \equiv \sum_\omega \left|\frac{\alpha_\omega}{\rho u_j}\right|^{v_j}$$

The three methods

BP

$$\min ||\alpha'||_1 \text{ s.t. } ||y - \Theta\,\alpha'||_2 \leq \epsilon$$

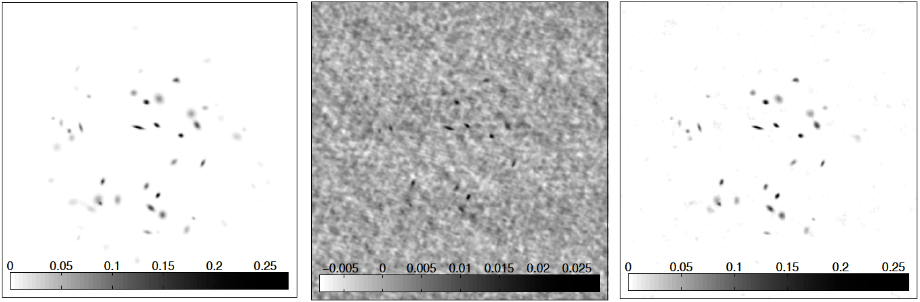
BP_+

$$\min ||\bar{x}'||_1 \text{ s.t. } y = \bar{\Phi}_{ri}\,\bar{x}' \text{ and } \bar{x}' \geq 0.$$

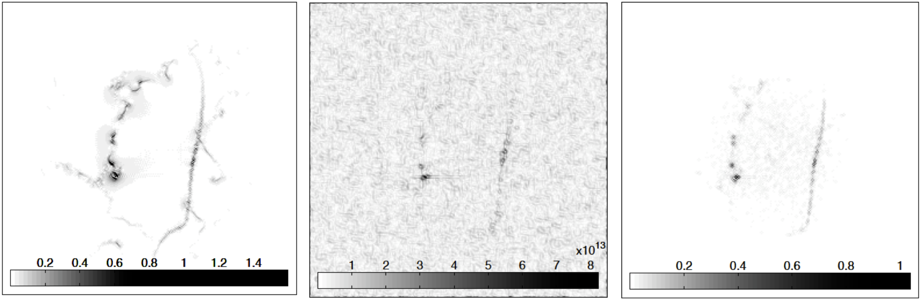
SBP

$$\min ||\alpha'||_s \text{ s.t. } ||\bar{y} - W_{cmb}\Phi_{ri}\Psi_s\alpha'||_2 \leq \epsilon$$

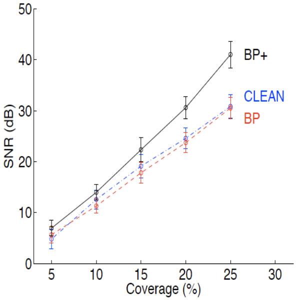
Compact Sources

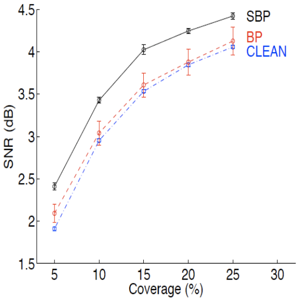


Cosmic Strings



SNR


50
40
30
20
10
0
SNR (dB)
5
10
15
20
25
30
Coverage (%)
BP+
CLEAN
BP


4.5
4
3.5
3
2.5
2
1.5
SNR (dB)
5
10
15
20
25
30
Coverage (%)
SBP
BP
CLEAN

Conclusions

- A new framework for image reconstruction in interferometry
- Simple BP provides the same image fidelity as CLEAN
- BP is more rapid than CLEAN in terms of no. iterations and computation time
- Prior statistical knowledge of the signal can greatly improve the reconstruction