Phase Error of Kim's Differential Atmospheric Phase Monitor
if the SNR is Unity or Greater
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The (real) signal voltage from the Satellite which is picked up by either antenna and band limited within the receiving system can be represented by the narrow band form $S(t)=s(t)\cdot\cos(\omega t-\phi(t))$ where s(t) and $\phi(t)$ represent the amplitude and phase and vary at a rate of the inverse bandwidth. Similarly the noise voltage from an antenna can be represented as $N(t)=n_1(t)\cdot\cos(\omega t-\phi_1(t))$. If we assume that the delays have been set so that the satellite is at the cneter of the white light fringe, the output of I from the I/Q demodulator will be:

$$I\propto\int_0^T[S(t)+N_1(t)]\cdot[S(t)+N_2(t)]\,dt$$

$$=\int_0^T[S^2(t)+S(t)\cdot(N_1(t)+N_2(t))+N_1(t)\cdot N_2(t)]\,dt$$

$$\begin{aligned}=\int_0^T[&s^2(t)\cdot\cos^2(\omega t-\phi(t))\\&+s(t)\cdot\cos(\omega t-\phi(t))\cdot(n_1(t)\cdot\cos(\omega t-\phi_1(t))+n_2(t)\cdot\cos(\omega t-\phi_2(t)))\\&+n_1(t)\cdot\cos(\omega t-\phi_1(t))+n_2(t)\cdot\cos(\omega t-\phi_2(t))]\end{aligned}$$

Only the $S^2(t)$ term integrates coherently, and the noise from the other terms will be small and may be ignored, so:

$$I\propto s^2$$

where s now refers to the rms of s(t).

The Q output will be similar:

$$\begin{aligned}Q\propto\int_0^T[&s^2(t)\cdot\cos(\omega t-\phi(t))\cdot\sin(\omega t-\phi(t))\\&+s(t)\cdot\cos(\omega t-\phi(t))\cdot(n_1(t)\cdot\cos(\omega t-\phi_1(t))+n_2(t)\cdot\sin(\omega t-\phi_2(t)))\\&+n_1(t)\cdot\cos(\omega t-\phi_1(t))+n_2(t)\cdot\sin(\omega t-\phi_2(t))]\,dt\end{aligned}$$

except that the $S^2(t)$ term will integrate to zero since one of the cosines has been converted to a sine. The remaining terms will integrate to approximately zero and we can apply the usual radiometer formula to dtermine the rms deviation of Q from zero:

$$Qrms \propto \sqrt{\frac{2s^2 \cdot n^2 + n^4}{B\tau}}$$

where s and n refer to rms values and the two noise powers have been assumed to be equal. B is the IF bandwidth. The noise in fringe phase angle measurement (radians) is Qrms / I:

$$\Phi\, rms = Qrms/I = \sqrt{\frac{2n^2/s^2 + n^4/s^4}{B\tau}}$$

In the usual radio astronomy case, the noise is much larger than the signal and this formula reduces to:

$$\Phi\, rms \doteq \frac{n^2}{s^2 \cdot \sqrt{B\tau}}$$

which is the usual ratio of noise power to signal power divided by $\sqrt{B\tau}$.

Kim calculates a signal to noise ratio of 5 dB ($n^2/s^2 = .3126$)for one satellite using a 1.6m dish. He gives $\sqrt{B\tau}$ as 4582 when sampling the phase at 20 Hz. Using the formula above and those conditions, $\Phi\, rms = 0.0107$ deg.