Atmospheric Refractive Signal Bending and Propagation Delay

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August 16, 2009

Contents

1 Recommendations

1. *Refractive Bending Calculation:* Use the Auer & Standish (2000) method (Equation 9) with the procedure described in §2.2. The refractivity ($N(P,T)$) is derived from the atmospheric model program ATM.

2. *Refractive Delay Calculation:* Use Equation 56 with refractivity derived from ATM. If an interim first-cut solution is desired, use Equation 61 with coefficients a, b, and c derived from Niell (1996).

2 Refractive Bending

The following derivation of a generalized refractive bending calculation using a simple ray-trace analysis was originally proposed by Auer & Standish (1979) and further developed by Hohenkerk & Sinclair (1985) and described in The Explanatory Supplement to the Astronomical Almanac (1992). A modern description of the algorithm can be found in Auer & Standish (2000). The slalib routine *slaRefro* uses a modified version of the Hohenkerk & Sinclair (1985) development of the Auer & Standish (1979) algorithm. *slaRefro* includes an atmospheric model (Liebe *et al.* (1993)) which allows for calculation of the atmospheric refractivity up to frequencies of 1 THz.

For a spherically-symmetric atmosphere, the astronomical refraction R is given by

$$R = \int_1^{n_0} \frac{\tan(z)}{n} dn \tag{1}$$

where n is the index of refraction, z is the zenith angle, and the integral is carried out along the path of the signal. From Snell's law,

$$nr\sin(z) = n_0 r_0 \sin(z_0), \tag{2}$$

in which r is the geocentric distance to a point in the atmosphere and n_0, r_0, and z_0 are the measured refractive index, geocentric distance, and source zenith angle at the location of the observer. In principle, R could be calculated directly from Equation 1 by numerical quadrature. But, as Auer & Standish (1979, 2000) point out, numerical difficulties at $z = 90$ make it preferable to use z itself as the variable of integration.

Auer & Standish (2000) derive a transformed version of Equation 1 which varies slowly over z and avoids the numerical difficulties at $z = 90$. Equation 1 can be written in terms of $\ln(n)$ as follows:

$$R = \int_0^{\ln(n_0)} \tan(z) d(\ln(n)) \tag{3}$$

Taking the logarithmic derivative of Equation 2

$$\begin{aligned} \ln(rn) &= \ln(n_0 r_0 \sin(z_0)) - \ln(\sin(z)) \\ \frac{d(\ln(rn))}{dz} &= -\frac{1}{\tan(z)} \end{aligned} \tag{4}$$

and substituting this expression into Equation 3 we find that

$$R = -\int_0^{\ln(n_0)} \frac{dz}{d(\ln(rn))} d(\ln(n)) \tag{5}$$

Further substituting the following

$$\begin{aligned} d(\ln(rn)) &= d(\ln(r)) + d(\ln(n)) \\ R(\ln(n_0)) &= R(z_0) \end{aligned} \tag{6}$$

we find that

$$\begin{aligned} R &= -\int_0^{z_0} \frac{d(\ln(n))}{d(\ln(r)) + d(\ln(n))} dz \\ &= -\int_0^{z_0} \frac{\frac{d(\ln(n))}{d(\ln(r))}}{1 + \frac{d(\ln(n))}{d(\ln(r))}} dz \end{aligned} \tag{7}$$

After substitution of the following

$$\frac{d(\ln(n))}{d(\ln(r))} = \frac{r}{n}\frac{dn}{dr} \tag{8}$$

we get the following

$$\boxed{R = -\int_0^{z_0} \frac{r\frac{dn}{dr}}{n + r\frac{dn}{dr}} dz} \tag{9}$$

Equation 9 is well-behaved at $z = 90°$ and can be evaluated by quadrature using equal steps in z. At each step in z the corresponding values for r, n, and $\frac{dn}{dr}$ must also be calculated. Solving Equation 2 for r

$$F(r) = nr - \frac{n_0 r_0 \sin(z_0)}{\sin(z)} = 0 \tag{10}$$

one can find the root of Equation 10 by Newton-Raphson iteration

$$r_{i+1} = r_i - \frac{F(r_i)}{F'(r_i)} \tag{11}$$

where

$$F'(r) = \frac{dn}{dr} r + n \tag{12}$$

Equation 9 is the refraction equation used in The Explanatory Supplement to the Astronomical Almanac (1992), Equation 3.281-1. A simple two component model of the atmosphere is often assumed. In this model, there is a discontinuity in $\frac{dn}{dr}$ at the tropopause, so the refraction integral must be calculated in two parts: one for the troposphere and another for the stratosphere. Note also that atmospheric inhomogeneities can be accounted for in this formalism by using multiple components in the integration.

2.1 Atmospheric Model

A simple model for the Earth's atmosphere defines it as follows:

- Spherically symmetric distribution of density with two layers (troposphere and stratosphere).
- Hydrostatic equilibrium.
- Perfect gas law applies.
- Temperature decreasing at a constant rate with height in the troposphere and constant in the stratosphere.
- The Gladstone-Dale relation, $n-1 = a\rho$, which relates the refractive index n and the density ρ, where a is a constant which depends only on the local physical properties of the atmosphere.
- Two layer structure with $a < \infty$ for $r_e \leq r \leq h_t$ and $a = \infty$ for $h_t \leq r \leq h_s$.
- Constant relative humidity in the troposphere which is equal to the relative humidity measured at the observer.
- Refractivity at the observer (N_0) and refractive index (n) related by: $n+1 = 10^{-6}N_0$, where N_0 is a function of the atmospheric pressure (P_0), temperature (T_0), and relative humidity (RH_0) at the observer.
- The following constants:
  - Universal gas constant: $R_g = 8314.32\ J/(mole * K)$
  - Molecular weight of dry air: $M_d = 28.9644\ gm/mole$
  - Molecular weight of wet air: $M_w = 18.0152\ gm/mole$
  - Molecular weight of atmosphere (mixture of dry and wet air): M_{atm}
  - Acceleration due to gravity at the center of mass of the vertical column of air above the observer at observer height h_0: g_m. See Appendix B for further details on the preferred expression for g_m.
  - Height of the Earth's geoid (assuming WGX84 spheroid) as a function of latitude: $r_{WGS84} = 6378.137\left(1 - \frac{\sin^2(\phi)}{298.257223563}\right)\ km$
  - Height of the observer above the geoid: h_0
  - Height of the troposphere above the geoid: h_t
  - Height of the stratosphere above the geoid: h_s
  - Total height of the observer: $r_0 = r_{WGS84} + h_0$
  - Total height of the troposphere: $r_t = r_{WGS84} + h_t$
  - Total height of the stratosphere: $r_s = r_{WGS84} + h_s$

Equation 9 requires a description of the radial variation of n and its derivative $\frac{dn}{dr}$, which depend upon the radial variation of P, T, and RH. A number of analytical expressions for $n(r)$ and $\frac{dn}{dr}$ have been used in the past, including the piecewise polytropic model of Garfinkel (1944, 1967).

In the following we derive the radial variation of the temperature (T) and pressure (P).

2.1.1 Temperature Distribution

The distribution of temperature with r is defined as:

$$T(r) = T_0 + \alpha(r - r_0) \tag{13}$$

$$\frac{dT}{dr} = \alpha \tag{14}$$

In the following analysis of the pressure distribution we will use these temperature relations.

2.1.2 Pressure Distribution

In the following I derive the distribution of pressure with height above the observer. The algorithm I describe follows closely that presented by Sinclair (1982), Murray (1983), and Hohenkerk & Sinclair (1985).

Combining the ideal gas law:

$$P = \frac{\rho R_g T}{M_{atm}} \tag{15}$$

and the equation for hydrostatic equilibrium:

$$\frac{dP}{dr} = -g_m \rho \tag{16}$$

and the temperature distribution relation (Equation 13) we find that:

$$\frac{dP}{P} = -\frac{g_m M_{atm}}{\alpha R_g} \frac{dT}{T} \tag{17}$$

Integrating Equation 17 yields:

$$\begin{aligned}
\int \frac{dP}{P} &= -\frac{g_m M_{atm}}{\alpha R_g} \int \frac{dT}{T} \\
\ln\left(\frac{P}{P_0}\right) &= \ln\left(\frac{T}{T_0}\right)^{-\frac{g_m M_{atm}}{\alpha R_g}} \\
\frac{P}{P_0} &= \left(\frac{T}{T_0}\right)^{-\frac{g_m M_{atm}}{\alpha R_g}} \\
&= \left(\frac{T}{T_0}\right)^{\beta} \qquad (18)
\end{aligned}$$

where I have defined:

$$\beta \equiv -\frac{g_m M_{atm}}{\alpha R_g} \tag{19}$$

The total atmospheric pressure (P) and density (ρ) are composed of two components: the partial pressure and density due to dry air (P_d and ρ_d) and the partial pressure and density due to wet air (P_w and ρ_w). Since the water vapour pressure P_w decreases much more rapidly than the total pressure P, we need to separate P into its constituent parts. These pressures and densities are related as follows:

$$P = P_d + P_w \tag{20}$$
$$\rho = \rho_d + \rho_w \tag{21}$$

using the Ideal Gas Law (Equation 15) for each component (dry, wet, and total), we can write Equation 15 as:

$$\begin{aligned} P &= \frac{R_g T}{M_{atm}} (\rho_d + \rho_w) \\ &= \frac{P_d M_d + P_w M_w}{M_{atm}} \end{aligned} \tag{22}$$

which allows us to write M_{atm} in terms of its dry and wet components as:

$$\begin{aligned} M_{atm} &= \frac{P_d M_d + P_w M_w}{P} \\ &= M_d - \frac{P_w (M_d - M_w)}{P} \end{aligned} \tag{23}$$

Combining Equations 23, 17, and 18 produces a general expression which describes the variation of P with r:

$$\begin{aligned} \frac{dP}{P} &= \frac{-g_m M_d}{\alpha R_g} \frac{dT}{T} + \frac{g_m M_d P_w}{\alpha R_g P_0} \left(\frac{T}{T_0}\right)^{-\beta} \left(1 - \frac{M_w}{M_d}\right) \frac{dT}{T} \\ &= \beta \frac{dT}{T} - \beta \frac{P_w}{P_0} \left(\frac{T}{T_0}\right)^{-\beta} \left(1 - \frac{M_w}{M_d}\right) \frac{dT}{T} \end{aligned} \tag{24}$$

Note that in Equation 24 g_m (Equation 87) and T (Equation 13) are known functions of r. Only the radial dependence of P_w is as yet unknown.

At this point we need to take a little diversion into the relationship between relative humidity (RH) and saturation vapor pressure (e_{sat}). In Appendix C we note that the approximation:

$$\frac{e_{sat}(P,T)}{e_{sat}(P_0,T_0)} = \left(\frac{T}{T_0}\right)^{\gamma} \tag{25}$$

for saturation vapor pressure agrees with the more exact expression (Equation 95: Buck (1981)) to within ±0.2 mb over the range P = 600 – 1200 mb and T = −30 – +20 C. Therefore, using Equation 25 in Equation 24 yields:

$$\frac{dP}{P} = \beta\frac{dT}{T} - \beta\frac{P_{w0}}{P_0}\left(\frac{T}{T_0}\right)^{\gamma-\beta}\left(1-\frac{M_w}{M_d}\right)\frac{dT}{T} \tag{26}$$

Integrating Equation 26 in the same way that was done for Equation 17 leads to the general expression which describes the radial dependence of atmospheric pressure:

$$\begin{aligned} \ln\left(\frac{P}{P_0}\right) &= \ln\left(\frac{T}{T_0}\right)^{\beta} + \frac{\beta}{\gamma-\beta}\left(1-\frac{M_w}{M_d}\right)\frac{P_{w0}}{P_0}\left[1-\left(\frac{T}{T_0}\right)^{\gamma-\beta}\right] \\ \frac{P}{P_0} &= \left(\frac{T}{T_0}\right)^{\beta}\exp(W) \end{aligned} \tag{27}$$

where we have defined:

$$W \equiv \frac{\beta}{\gamma-\beta}\left(1-\frac{M_w}{M_d}\right)\frac{P_{w0}}{P_0}\left[1-\left(\frac{T}{T_0}\right)^{\gamma-\beta}\right] \tag{28}$$

Sinclair (1982) points out that $W \lesssim 0.003$, which allows one to expand the exponential as $\exp(W) \simeq 1 + W$ and write Equation 27 as:

$$\boxed{\frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{\beta} + \frac{\beta}{\gamma-\beta}\left(1-\frac{M_w}{M_d}\right)\frac{P_{w0}}{P_0}\left[\left(\frac{T}{T_0}\right)^{\beta} - \left(\frac{T}{T_0}\right)^{\gamma}\right]} \tag{29}$$

2.1.3 Atmospheric Radio/Submillimeter Refractivity

It would be convenient at this point to develop and expression for the atmospheric refractivity N at radio and submillimeter wavelengths[1] as functions of P and T to use in Equation 9. In general, the refractivity of moist air at microwave frequencies depends upon the permanent and induced dipole moments of the molecular species that make up the atmosphere. The primary species that make up the dry atmosphere, nitrogen and oxygen, do not have permanent dipole moments, so contribute to the refractivity via their induced dipole moments. Water vapour does have a permanent dipole moment. Permanent dipole moments contribute to the refractivity as $N \propto \frac{P}{T^2}$, while induced dipole moments contribute as $N \propto \frac{P}{T}$, where P is the pressure and T is the temperature of the species.

A simple parameterization of the frequency-independent (nondispersive) refractivity at the zenith is given by the Smith-Weintraub equation (Smith & Weintraub, 1953):

$$N = k_1\frac{P_d}{T} + k_2\frac{P_w}{T} + k_3\frac{P_w}{T^2} + k_4\frac{P_c}{T} \tag{30}$$

[1] For a brief description of atmospheric refractivity at optical wavelengths, see Appendix A

where P_d, P_w, and P_c are the partial pressures due to dry air, water vapour, and carbon dioxide, T is the temperature of the atmosphere, and k_1, k_2, k_3, and k_4 are constants. The dry and wet air refractivities are then given by:

$$N_d = k_1 \frac{P_d}{T} \tag{31}$$

$$N_w = k_2 \frac{P_w}{T} + k_3 \frac{P_w}{T^2} \tag{32}$$

$$N_d = k_4 \frac{P_c}{T} = \frac{5}{3} \frac{P_c}{T} \tag{33}$$

Since the partial pressure due to carbon dioxide is $\sim 0.03\%$ of the total pressure, this term is often ignored or lumped into the dry air contribution in the simple parameterizations of atmospheric refractivity.

The dry air contribution to this refractivity (N_d) is primarily due to oxygen and nitrogen, and is nearly in hydrostatic equilibrium. Therefore, N_d does not depend upon the detailed behaviour of dry air pressure and temperature along the path through the atmosphere, and can be derived based on local atmospheric temperature and pressure measurements. The wet air refractivity (N_w) can be inferred from local water vapour radiometry measurements.

Closed-form approximations for the nondispersive $N(P,T)$ have been derived for use at frequencies below 100 GHz by Brussaard & Watson (1995):

$$\begin{aligned} {}^{BW}N &= 77.6 \frac{P_d}{T} + 72.0 \frac{P_w}{T} + 3.75 \times 10^5 \frac{P_w}{T^2} \quad ppm \\ &= 77.6 \frac{P}{T} - 5.6 \frac{P_w}{T} + 3.75 \times 10^5 \frac{P_w}{T^2} \quad ppm \end{aligned} \tag{34}$$

and Smith & Weintraub (1953) (see also Crane (1976) and Liebe & Hopponen (1977)):

$$\begin{aligned} {}^{SW}N &= 77.6 \frac{P_d}{T} + 72.0 \frac{P_w}{T} + 3.776 \times 10^5 \frac{P_w}{T^2} \quad ppm \\ &= 77.6 \frac{P}{T} - 12.8 \frac{P_w}{T} + 3.776 \times 10^5 \frac{P_w}{T^2} \quad ppm \end{aligned} \tag{35}$$

where

P_d is the partial pressure of dry gases in the atmosphere (in mb),

P_w is the partial pressure of water vapor (in mb),

P is the total barometric pressure (in mb), which is equal to $P_d + P_w$, and

T is the ambient air temperature (in Kelvin).

The best of the closed-form approximations to the nondispersive refractivity, though, is the equation derived by Rüeger (2002) which uses what he describes as the "best average" values for the coefficients k_1, k_2, and k_3 (which includes a 375 ppm contribution due to carbon dioxide in the k_1 term):

$$\begin{aligned} {}^{Rueger}N &= 77.6890\frac{P_d}{T} + 71.2952\frac{P_w}{T} + 3.75463 \times 10^5 \frac{P_w}{T^2} \quad ppm \\ &= 77.6890\frac{P}{T} - 6.3938\frac{P_w}{T} + 3.75463 \times 10^5 \frac{P_w}{T^2} \quad ppm \end{aligned} \tag{36}$$

Comparing these three closed-form expressions for radio refractivity at representative values of pressure (550 mB; appropriate for the ALMA site) and site altitude (5.5 km; appropriate for the ALMA site) to a more exact model of the atmospheric refractivity (which includes a dispersive contribution), we find that:

- The Brussaard & Watson (1995) and Smith & Weintraub (1953) expressions agree to within less than 0.5% for all calculations.
- The Brussaard & Watson (1995) and Smith & Weintraub (1953) expressions agree with a more exact (*i.e.* including dispersive refractivity; Liebe (1989)) atmospheric model prediction of N to better than:
  - 3% at 8 and 230 GHz
  - 6% at 370 GHz (this is a band edge for ALMA)
  - 7% at 950 GHz (the highest band edge for ALMA)
- The Rüeger (2002) and Liebe (1989) model predictions agree to better than:
  - 0.12% at 8 GHz
  - 0.7% at 230 GHz
  - 3.5% at 370 GHz
  - 5.5% at 950 GHz

Note that the closed-form expressions described above are only good for calculations at frequencies far from telluric lines. For general high-accuracy calculations at submillimeter wavelengths one must use an atmospheric model (such as Liebe (1989) or Liebe *et al.* (1993)) which incorporates both nondispersive and dispersive contributions to the refractivity to derive the total atmospheric refractivity. For ALMA, ATM is the model of choice.

2.1.4 Application to the Troposphere and Stratosphere

In the following we investigate the parametric forms for $P(r)$, $T(r)$, $RH(r)$, n, and $\frac{dn}{dr}$ in the troposphere and the stratosphere:

Troposphere: $(r_e \leq r \leq h_t)$

$$T(r) = T_0 + \alpha(r - r_0) \tag{37}$$

$$P(r) = P_0 \left(\frac{T}{T_0}\right)^{\beta} + \frac{\beta P_{w0}}{\gamma - \beta}\left(1 - \frac{M_w}{M_d}\right)\left[\left(\frac{T}{T_0}\right)^{\beta} - \left(\frac{T}{T_0}\right)^{\gamma}\right] \tag{38}$$

$$RH(r) = RH_0 \text{ (constant)} \tag{39}$$

$$n = 1 + 10^{-6}N(r) \tag{40}$$

$$\frac{dn}{dr} = 10^{-6}\frac{dN(r)}{dr} \tag{41}$$

Stratosphere: $(h_t \leq r \leq h_s)$

For isothermal atmospheric layers (like the stratosphere), $\alpha = 0$ and we use the approximation $\ln(1+\epsilon) \rightarrow \epsilon$ as $\epsilon \rightarrow 0$, which makes Equations 13 and 18 become

$$T(r) = T(h_t) \text{ (constant)} \tag{42}$$

$$P(r) = P(h_t) \exp\left[\frac{g_m M_{atm}(r - r_t)}{R_g T(h_t)}\right] \tag{43}$$

$$RH(r) = 0 \tag{44}$$

$$\begin{aligned} n &= 1 + (n(h_t) - 1) \exp\left[\frac{g_m M_{atm}(r - r_t)}{R_g T(h_t)}\right] \\ &= 1 + 10^{-6}N(h_t) \exp\left[\frac{g_m M_{atm}(r - r_t)}{R_g T(h_t)}\right] \end{aligned} \tag{45}$$

$$\begin{aligned} \frac{dn}{dr} &= -\frac{g_m M_{atm}(r - r_t)}{R_g T(h_t)}(n(r_t) - 1) \exp\left[\frac{g_m M_{atm}(r - r_t)}{R_g T(r_t)}\right] \\ &= -\frac{g_m M_{atm}(r - r_t)}{R_g T(h_t)} 10^{-6}N(r_t) \exp\left[\frac{g_m M_{atm}(r - r_t)}{R_g T(r_t)}\right] \end{aligned} \tag{46}$$

Figure 1 shows the situation.

2.2 The Procedure

As noted above, the Garfinkel (1944, 1967) model of the atmosphere contains a discontinuity in the temperature gradient at the tropopause, which results in a discontinuity in the integrand of Equation 9. It is therefore necessary to evaluate the integrand in two steps: one for the troposphere and a second for the stratosphere.

To calculate the refraction using Equation 9 for each layer (troposphere and stratosphere, for the atmospheric model used here), follow these steps:

1. Calculate r at each step in zenith distance by solving Equation 2 using Newton-Raphson iteration

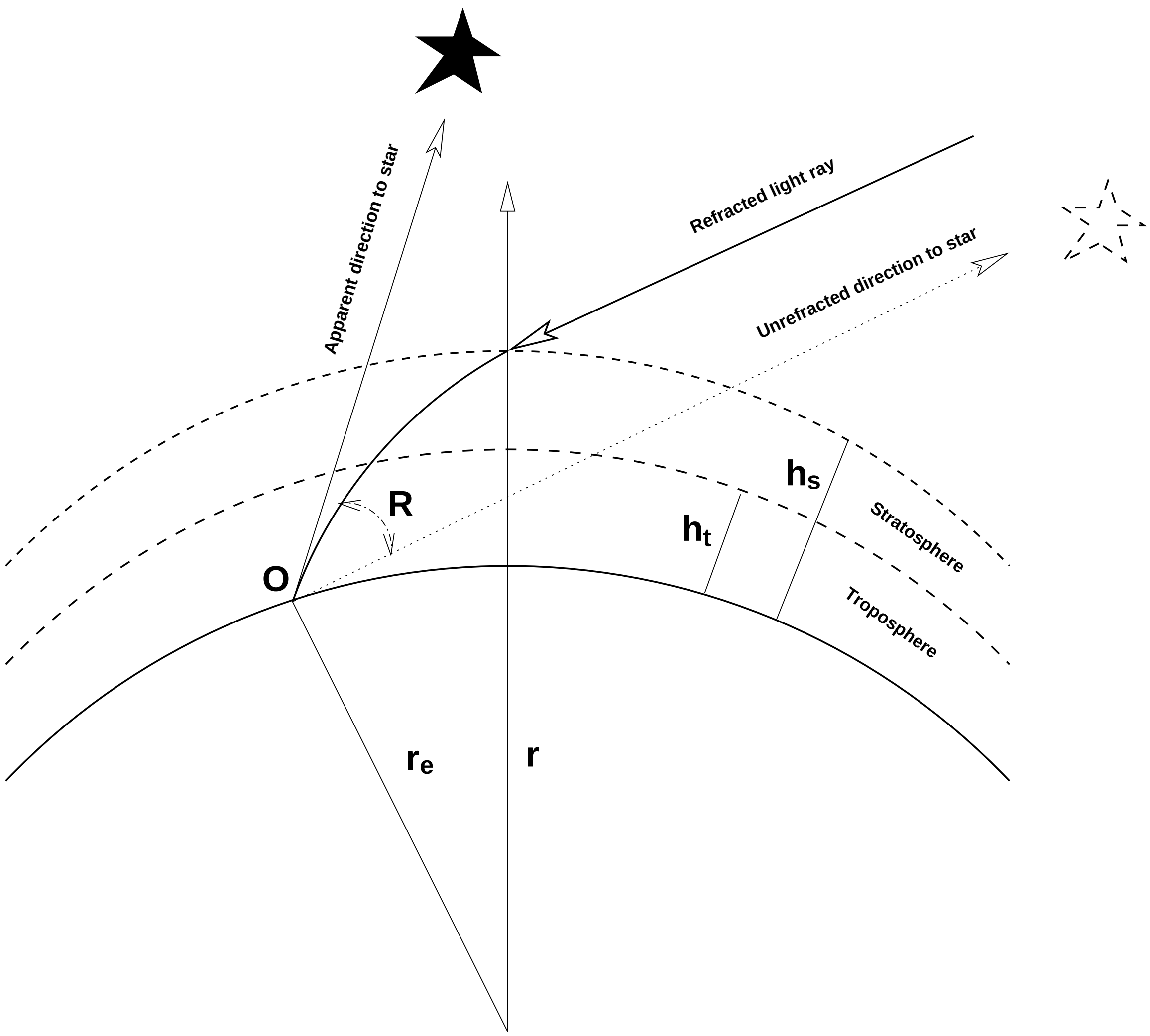

Figure 1: Drawing of simple refraction. Based on the very similar drawing shown in Hohenkerk & Sinclair (1985).

$$
\begin{aligned}
r_{i+1} &= r_i - \frac{F(r_i)}{F'(r_i)} \\
&= r_i - \left[\frac{n_i r_i - n_0 r_0 \frac{\sin(z_0)}{\sin(z)}}{n_i + r_i \frac{dn_i}{dr_i}} \right]
\end{aligned} \tag{47}
$$

for $i = 1, 2, \ldots$, and where r_1 is the value of r calculated at the previous step of the integration. Convergence of this iteration is fast, requiring only about 4 steps.

2. Once you have a converged solution for r, calculate n and $\frac{dn}{dr}$ using your chosen atmospheric model.

3. Integrate Equation 9 over each interval (troposphere and stratosphere) using Simpson's rule with summation over equal steps in z

$$
\int_{r_0}^{r_3} f(r)dr = \frac{\Delta r}{3} \left(f_0 + 4f_1 + 2f_2 + f_3 \right) \tag{48}
$$

where f_n is $f(x)$ evaluated at $x = x_0, x_1, x_2,$ and x_3.

4. Compare each integration result to the result of the previous step of this integration. Check for either convergence (*slaRefro* uses $|\int f(z_i)dz - \int f(z_{i-1})dz| \leq 10^{-8}$) or maximum iteration reached (*slaRefro* uses 16384). If convergence or maximum iteration not reached, go to step 1.

2.3 Alternate Routes for Refractive Bending Calculation

Instead of doing the integral Equation 9, various approximations are often made to reduce this expression to a simple analytical form. Some of the more generally useful forms are based on a generator function formalism which assumes an exponential atmospheric profile

$$
N(h) = N_0 \exp\left[-\frac{(r - r_0)}{H} \right] \tag{49}
$$

where r and r_0 are height coordinates and H is the effective height of the atmosphere

$$
H = \frac{R_g T}{M_{atm} g_m} \tag{50}
$$

where R_g is the universal gas constant, M_{atm} is the molar mass of the atmosphere, T is the temperature of the atmosphere, and g_m is the gravitational acceleration constant measured at the center of the vertical column of air (see §2.1).

One form of this generator function formalism has been described by Yan & Ping (1995) and Yan (1996) as follows:

$$
R(E) = R_0 \cos(E) m'(E) \tag{51}
$$

where

$$m'(E) = \frac{1}{\sin(E) + \frac{A_1}{I^2 \csc(E) + \frac{A_2}{\sin(E) + \frac{13.24969}{I^2 \csc(E) + 173.4233}}}} \tag{52}$$

and

$$I = \sqrt{\frac{r_0}{2H}} \tan(E) \tag{53}$$

See ALMA memo 366 for further information on the use of this formalism for calculating the refraction. Note, though, that the analysis presented in Yan & Ping (1995) purports to an accuracy far better than is realistic. Furthermore, comparison to the "gold standard" refraction calculation done by the SLALIB routine *sla_refro* suggests that the parametric equation presented in Yan & Ping (1995) is tuned to a specific set of site and metrological conditions (sea level and relatively dry).

3 Refractive Delay Due to the Atmosphere

The calculation of the atmospheric refractive delay parallels that for refractive bending. Historically, a number of "delay models" have been developed to calculate refractive delay. In the following I give an overview of these atmospheric refractive delay models.

The delay experienced by an incoming signal due to its propagation through the Earth's atmosphere is given by:

$$\tau_{atm} = \int_s (n-1)\, ds \tag{54}$$

where s is the path through and n is the refractive index of the atmosphere. Since n is very nearly unity for the Earth's atmosphere one normally uses the "refractivity" (N) instead of the index of refraction. Refractivity and refractive index are related as follows:

$$N = 10^6 (n-1) \tag{55}$$

For measurements from the surface of the Earth along a given zenith angle z, Equation 54 becomes:

$$\boxed{\tau_{atm} = \int_{r_0}^{\infty} \frac{10^{-6} N(r)}{\cos(z)} dr} \tag{56}$$

In practice the upper limit to the integral in Equation 56 is the top of the stratosphere. By using an atmospheric model to calculate $N(r)$ one can numerically integrate Equation 56 to derive the refractive delay due to the atmosphere.

3.1 Alternate Routes for Refractive Delay Calculation

In lieu of an atmospheric model based calculation of N one can separate the atmospheric delay into contributions due to the dry and wet atmosphere:

$$\tau_{atm} = \tau_d + \tau_w \tag{57}$$

where τ_d is the contribution due to dry air while τ_w is the contribution due to wet air. In general τ_d and τ_w are parameterized in terms of a zenith contribution to the delay which is dependent upon local atmospheric conditions (Z) and a "mapping function" (M) which relate delays at an arbitrary elevation angle E to that at the zenith:

$$\begin{aligned} \tau_{atm} &= ZM \\ &= Z_d M_d + Z_w M_w \end{aligned} \tag{58}$$

Since the elevation angle E is the unrefracted source elevation, refraction effects are included in the mapping functions M. In the following I describe calculations of Z and M.

3.1.1 Zenith Delay

The contribution to the atmospheric delay at the zenith (Z) is a measure of the integrated refractivity of the atmosphere at the zenith (N). As was noted in §2.1.3 there are closed-form expressions for $N(P, T)$ which are appropriate for calculations at frequencies below 100 GHz. For high-frequency calculations, one must use an atmospheric model.

3.1.2 Mapping Functions

The simplest form for the mapping function (M), which relates the delay at an arbitrary elevation angle E to that at the zenith, is given by the plane-parallel approximation for the Earth's atmosphere:

$$M = \frac{1}{\sin(E)} \tag{59}$$

This simple form is in fact inadequate, which led Marini (1972) to consider corrections to this simple functional form which accounted for the Earth's curvature. Assuming an exponential atmospheric profile where the atmospheric refractivity varies exponentially with height above the antenna, Marini (1972) developed a continued fraction form for the mapping function:

$$M = \frac{1}{\sin(E) + \frac{a}{\sin(E) + \frac{b}{\sin(E) + c}}} \tag{60}$$

where I include only the first three terms in the continued fraction. Two slight modifications to the Marini (1972) continued fraction functional form can be implemented to force $M = 1$ at the zenith:

- Normalize Equation 60 as follows:

$$M = \frac{1 + \frac{a}{1+\frac{b}{1+c}}}{\sin(E) + \frac{a}{\sin(E)+\frac{b}{\sin(E)+c}}} \tag{61}$$

  See Niell (1996) for a discussion of how to use this form of the mapping function[2], including derivation of the coefficients a, b, and c.

- Replace the even numbered $\sin(E)$ terms (*i.e.* the second, fourth, sixth, etc.) with $\tan(E)$:

$$M = \frac{1}{\sin(E) + \frac{a}{\tan(E)+\frac{b}{\sin(E)+c}}} \tag{62}$$

  Chao (1974) introduced this modification by truncating the Marini (1972) form to include only two terms.

A more generalized continued-fractional form for the mapping function was developed by Yan & Ping (1995):

$$M = \frac{1}{\sin(E) + \frac{a}{I^2\tan(E)+\frac{b}{\sin(E)+\frac{c}{I^2\tan(E)+d}}}} \tag{63}$$

where

$$I = \sqrt{\frac{r_0 \sec(E)}{2H}} \tag{64}$$

is the "normalized effective zenith argument" of function which includes the "normalized effective height" of the atmosphere (H) defined as:

$$H = \frac{1}{N_0}\int_{h_0}^{\infty} N(h)dh \tag{65}$$

For an exponentially-distributed atmosphere:

$$N(h) = N_0 \exp\left(\frac{-(h-h_0)}{H}\right) \tag{66}$$

Normally, $h_0 = 0$ (*i.e.* start the integration from the ground).

The constants a, b, c, d, etc. in the continued fraction forms above are generally derived from analytic fits to ray-tracing results either for standard atmospheres or for observed atmospheric profiles based on radiosonde measurements. The mapping functions derived in Niell (1996) and Davis (1985) are derived in this way.

A physically more correct mapping function has been derived by Lanyi (1984). Unlike previous mapping functions, Lanyi does not fully separate the dry and wet contributions to the delay, which is a more physically correct approximation. It is based on an ideal

[2]Note that Equation 4 in Niell (1996) contains a typo. The numerator should be just A, rather than $\frac{1}{A}$. See Niell (2000). Equation 61 lists the correct form for this equation.

model atmosphere whose temperature is constant from the surface to the inversion layer h_1, then decreases linearly with height at rate W (the lapse rate) from h_1 to the tropopause height h_2, then is assumed to be constant above h_2. This mapping function is designed then to be a semi-analytic approximation to the atmospheric delay integral that retains an explicit temperature profile that can be determined using meteorological measurements. The mapping function is expanded as a second-order polynomial in Z_d and Z_w, plus the largest third-order term). It is nonlinear in Z_d and Z_w. It also contains terms which couple Z_d and Z_w, thus including terms which arise from the bending of the signal path through the atmosphere. The functional form for the atmospheric delay in this Lanyi (1984) model is given by:

$$\tau_{atm} = \frac{F(E)}{\sin(E)} \tag{67}$$

where

$$F(E) = F_d(E)Z_d + F_w(E)Z_w + \frac{F_{b1}(E)Z_d^2 + 2F_{b2}(E)Z_dZ_w + F_{b3}(E)Z_w^2}{\Delta} + \frac{F_{b4}(E)Z_d^3}{\Delta^2} \tag{68}$$

where Z_d = Dry atmospheric zenith delay, Z_w = Wet atmospheric zenith delay, F_{bn} = n-th bending contributions to the delay, Δ = Dry atmospheric scale height = $\frac{kT_0}{mg_c}$, k = Boltzmann's constant, T_0 = Daily average surface temperature, m = Mean molecular mass of dry air, and g_c = air column center of gravity gravitational acceleration. With standard values of k, m, $T_0 = 292K$ (appropriate for mid-latitudes), and $g_c = 978.37$ cm/s^2, $\Delta = 8.6$ km.

The dry, wet, and bending contributions are expressed in terms of moments of the refractivity. The bending terms are evaluated for the ideal model atmosphere and thus give the dependence of the delay on the four parameters T_0, W, h_1, and h_2. Therefore, the Lanyi (1984) model relies upon accurate surface meteorological measurements at the time of the observations to which the delay model is applied.

3.1.3 Mapping Function Summary

- Differences between the various mapping functions only show up at low elevation (< 10 degrees). Since geodesists do observe at very low elevations, these differences can be significant. This is not the case for astronomers.

- The Yan & Ping (1995) form has a "cousin" used for refractive bending ("refraction correction") calculations, making it a convenient choice for both refractive delay and bending calculations.

3.2 Antenna Height Correction to Total Atmospheric Delay

In the calculation of the zenith atmospheric delay at an antenna it is assumed that the atmospheric properties (P, T, RH) are the values measured at the focal plane of the antenna. For example, in VLBI each station has a set of associated weather measurements which are used to calculate Z. For a clustered array like the VLA or ALMA, the affect of the differences in antenna focal plane height above some reference point need to be accounted for.

For the VLA (not EVLA), CALC was not used to calculate the atmospheric delay. The antenna height correction was incorporated with a simple atmospheric delay correction by correcting for the path difference between each VLA antenna and a reference point at the center of the array. For the VLA case, the extra atmospheric path due to a difference in antenna height above the center-of-the-array reference point (ΔH, in ns) is given by:

$$\Delta T = \frac{10^{-6} N_0 \Delta H + T_z \frac{w}{\sin(E)}}{\sin(E)} \tag{69}$$

where N_0 is the atmospheric refractivity, T_z is the atmospheric zenith delay calculated using the VLA weather station (which is located near the center of the array), w is the geometric w of the antenna (in ns). The first term is the antenna height correction to the zenith delay, while the second term is a simple atmospheric delay correction. For EVLA, CALC will be used to calculate both geometric and atmospheric delay. We believe (though have not confirmed) that CALC also calculates the antenna height correction (first term in the equation above) given antenna heights relative to the reference point at the center of the array. ALMA will need to include this antenna height correction term.

A simple estimate of the magnitude of the antenna height difference correction at the zenith can be gotten by assuming that the pressure P changes linearly with height. Then 52 cm of additional antenna height (the current difference in height between the two ATF antennas) out of a total atmospheric height of 8 km would correspond to:

$$\frac{52\ cm}{8\ km} P = 0.068\ mb \tag{70}$$

where I have assumed P = 1053 mb. The dry term zenith atmospheric delay changes approximately like 2.3 mm/mb of pressure change. A pressure change of 0.068 mb corresponds to approximately 156 micron of path difference. This is consistent with alternate back-of-the-envelope calculations of this quantity.

3.3 Differential Excess Atmospheric Delay Between Two Antennas

NOTE: The following is just an aside. Since CALC or any other analysis of the atmospheric delay at an antenna calculates the total integrated delay along the path of observation, the differential delay between two antennas is accounted for in any differencing calculations done during baseline determination.

The differential delay induced in an interferometer by a horizontally stratified troposphere results from the difference in zenith angle of the source at the antennas. Thompson, Moran, and Swensen (2001) pp. 516-518 discuss the atmospheric delay induced along an interferometer baseline. The excess path length is given by:

$$L = 10^{-6} N_0 \int_0^\infty \exp\left(-\frac{h}{h_0}\right) dy \tag{71}$$

where N_0 is the refractivity at the Earth's surface, h is the height above the Earth's surface, h_0 is the atmospheric scale height, y is the length coordinate along the direction to the source, and E is the antenna elevation while observing the source. Note that refraction is neglected. One can relate y, h, h_0, and E as follows (see Figure 13-4 in Thompson, Moran, and Swensen (2001), page 517) using the cosine rule on the triangle formed by r_0, y, and $r_0 + h$:

$$(r_0 + h)^2 = r_0^2 + y_0^2 - 2r_0 y \cos(180 - z) \tag{72}$$

Solving for h and using elevation rather than zenith angle yields:

$$h = y \sin(E) + \frac{y^2 - h^2}{2r_0} \tag{73}$$

For the triangle which is formed by sides y, h, and the side which is equal to $y \sin(z_i)$, we can write:

$$y^2 - h^2 \simeq (y \sin(z_i))^2 \tag{74}$$

Since $r_0 \simeq 6370$ km and $h \simeq 8$ km (the height of the troposphere), $r_0 \gg h$. Since $z_i \simeq z + \frac{h}{r_0}$, $z_i \simeq z$. The equation for h in terms of y, E, and r_0 then becomes:

$$h \simeq y \sin(E) + \frac{y^2}{2r_0} \cos^2(E) \tag{75}$$

(Thanks to Dick Thompson for filling-in some of the details of this calculation).

We can now write the expression for L as follows:

$$L \simeq 10^{-6} N_0 \int_0^\infty \exp\left(-\frac{y}{h_0}\sin(E)\right) \exp\left(-\frac{y^2}{2r_0 h_0}\cos^2(E)\right) dy \tag{76}$$

Since $\frac{y}{r_0 h_0} \ll 1$, the second term in the equation above can be expanded with a Taylor series so that:

$$L \simeq 10^{-6} N_0 \int_0^\infty \exp\left(-\frac{y}{h_0}\sin(E)\right) \left(1 - \frac{y^2}{2r_0 h_0}\cos^2(E) + \frac{y^4}{8r_0^2 h_0^2}\cos^4(E) + ...\right) dy \tag{77}$$

Integration yields:

$$L \simeq 10^{-6} N_0 h_0 \csc(E) \left(1 - \frac{h_0}{r_0}\cot^2(E) + \frac{3h_0^2}{r_0^2}\cot^4() + ...\right) \tag{78}$$

Writing this equation in terms involving $\csc(E)$, the excess path length L becomes:

$$\begin{aligned} L \simeq\ & 10^{-6} N_0 h_0 \\ & \left[\left(1 + \frac{h_0}{r_0} + \frac{3h_0^2}{r_0^2}\right)\csc(E) - \left(\frac{h_0}{r_0} + \frac{6h_0^2}{r_0^2}\right)\csc^3(E) + \frac{3h_0^2}{r_0^2}\csc^5(E) + ...\right] \end{aligned} \tag{79}$$

Taking the derivative of L with respect to E and multiplying this derivative by the baseline length D divided by r_0 yields the atmospheric differential delay between two antennas separated by baseline D:

$$\frac{dL}{dE} \simeq \frac{-DN_0h_0\cot(E)}{10^3r_0} \left[\left(1+\frac{h_0}{r_0}+\frac{3h_0^2}{r_0^2}\right)\csc(E) - 3\left(\frac{h_0}{r_0}+\frac{6h_0^2}{r_0^2}\right)\csc^3(E) + \frac{15h_0^2}{r_0^2}\csc^5(E) + ...\right] \quad (80)$$

where D is in m, h_0 is in km, and r_0 is in km. Note that one must calculate N_0 using a suitable atmospheric model which uses measurements of the local atmospheric pressure, temperature and relative humidity to derive the resultant differential residual delay.

I have produced a plot of this relation as a function of N_0 for baseline lengths running from 10 to 100,000 m and elevation 1 to 90 degrees (Figure 2. Zooming in a bit on the vertical (baseline length) axis of this plot, I have produced two zoomed versions of Figure 2. One for baselines up to 1 km (Figure 3), and a second for baselines up to 100 m (Figure 4).

3.4 Some Background on References

In the following I give some background information on some of the references quoted in this section:

Niell (1996): *Global Mapping Functions for the Atmospheric Delay at Radio Wavelengths.* The standard reference for the derivation of a global mapping function for atmospheric delay. This derivation of the mapping function is really somewhat unique in that it attempts to analytically represent the global weather variations as a function of location (latitude) and time of year, and contains to adjustable parameters (i.e. does not require input pressure and temperature for each station). Note that Equation 4 in this paper has a typo whereby the terms which are printed as "1/term" in both the numerator and denominator should really be just "term" in both the numerator and denominator.

Davis (1985): *Geodesy by Radio Interferometry: Effects of Atmospheric Modeling Errors on Estimates of Baseline Length.* An application of a modified Smith-Weintraub refractivity and the Niell mapping functions.

Sovers *et al.* **(1998):** *Astrometry and Geodesy with Radio Interferometry: Experiments, Models, Results.* An excellent overview paper describing the details involved in calculating geometric and atmospheric delay. Uses the Lanyi (1984) model for the mapping function, which is a significant departure from the standard (*i.e.* Niell) mapping functions which derive from the Marini (1972) reduced fraction functional form.

Lanyi (1984): *Tropospheric Delay Effects in Radio Interferometry.* Derivation of a new "tropospheric" (really atmospheric) mapping function which, unlike previous mapping functions, takes account of second and third order effects in the refractivity

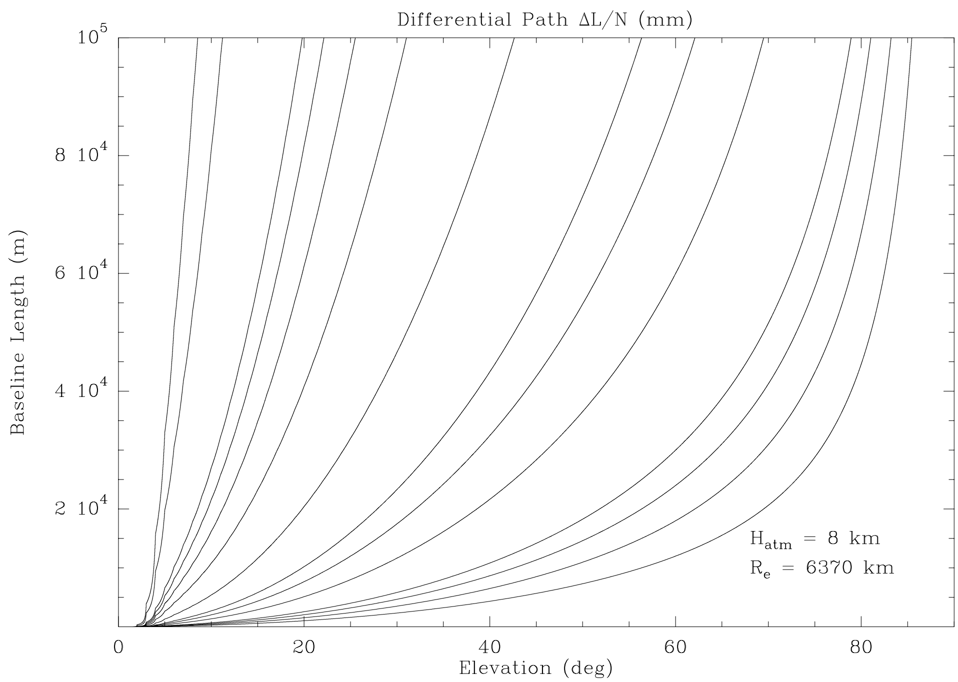

Figure 2: Plot of Equation 80 as a function of N_0 for baseline lengths running from 10 to 100,000 m and elevation 1 to 90 degrees. The contour levels on this plot are 10, 15, 20, 25, 50, 75, 100, 200, 400, 600, 800, 1000, 3000, and 5000 micron; right to left.

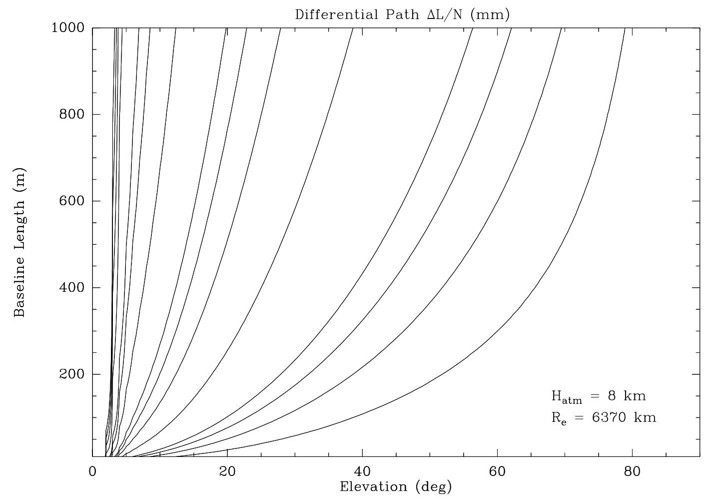

Figure 3: Same as Figure 2 but limited to baselines up to 1 km. The contour levels on this plot are 10, 15, 20, 25, 50, 75, 100, 200, 400, 600, 800, 1000, 3000, and 5000 micron; right to left.

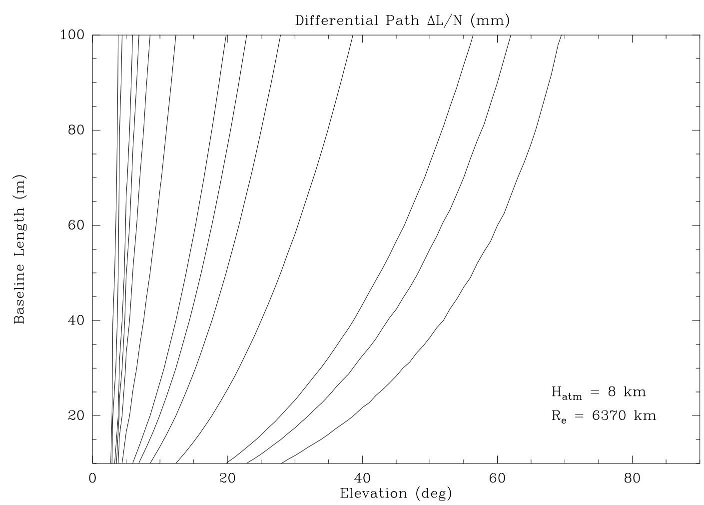

Figure 4: Same as Figure 2 but limited to baselines up to 100 m. The contour levels on this plot are 0.05, 0.075, 0.1, 0.25, 0.5, 0.75, 1.0, 2.5, 5.0, 7.5, 10, 25, 50 micron; right to left.

which are due to refractive bending. This derivation of the mapping function is really quite unique in that it does not fully separate the dry and wet contributions to the delay, making it a physically more exact representation. It is claimed to be more accurate than previous (i.e. Niell) mapping functions for E ¿ 4 degrees, and the error due to the derived analytic form for the mapping function is estimated to be less than 0.02% for E ¿ 6 degrees.

Yan & Ping (1995): *The Generator Function Method of the Tropospheric Refraction Corrections.* Another derivation of a new "tropospheric" (really atmospheric) mapping function. A cousin to existing reduced-fraction expansions of the mapping function.

Yan (1996): *A New Expression for Astronomical Refraction.* Related to the Yan & Ping (1995) reference above, but applied to the refraction calculation problem. Using the Yan & Ping (1995) and Yan (1996) references one can apply a unified formalism to both the atmosphere-induced refractive delay and bending problems.

A Atmospheric Optical Refractivity

Note: This is a slightly-modified version of Section 4.4.2 in ALMA Memo 366. Most importantly, I have corrected an error in the expression for N_{TP} in Livengood et al. (1999) reference.

Refractivity in the optical is cast in a slightly different form than that in the radio due to the fact that at optical wavelengths refractivity is no longer frequency independent. Birch & Downs (1993) (see also Livengood *et al.* (1999)) state that the optical refractivity is given by the following:

$$N_0^{opt} = N_{STP} \times N_{TP} - N_{RH} \tag{81}$$

where

$$\begin{aligned}
N_{STP} &= 83.4305 + \frac{24062.94}{130 - \lambda^{-2}} + \frac{159.99}{38.9 - \lambda^{-2}} && (82)\\
N_{TP} &= \frac{P_d}{1.01325 \times 10^3} \frac{(273.15 + 15)}{T} \frac{\left[1 + (3.25602 - 0.00972T)P_d \times 10^{-6}\right]}{1.00047} && (83)\\
N_{RH} &= P_w \times (37.345 - 0.401\lambda^{-2}) \times 10^{-3} && (84)
\end{aligned}$$

with P_d and P_w in mb, T in K, and λ in μm. Note that we have ignored the small correction for an increase in CO_2 concentration in Equation 81.

B Acceleration Due to Gravity

NOTE: Pat Wallace provided me with a nice summary of these references, which I reproduce and expand upon below.

The mean acceleration due to gravity (g_m) at the center of mass of a vertical column of air above an observer is given by:

$$g_m = \frac{\int_0^\infty dz\rho(z)g(z)}{\int_0^\infty dz\rho(z)} \tag{85}$$

By expanding $g(z)$ to first-order in z, fits to harmonic forms of g_m as a function of latitude (ϕ) can be derived.

Most references which derive the mean acceleration due to gravity at a given latitude calculate this quantity with reference to the center of mass of a vertical column of air above an observer (H_c), as stated above. It is often convenient to calculate g_m as a function of the height of an observer above sea level on the surface of the Earth (h_0). Saastamoinen (1972) points out that, due to the poleward slope of the tropopause and seasonal variations of T and P, regional and seasonal variations in H_c tend to be smoothed out. To an accuracy of ± 0.4 km, H_c and h_0 are related by:

$$H_c = 0.9\ h_0 + 7.3\ km \tag{86}$$

In the following I state a variety of formulations for g_m as functions of latitude (ϕ) and observer height above sea level (h_0, in km). Note that these expressions are for an observer in free air, and will be significantly different for an observer sitting on a slab of rock, which tends to be the case.

There are many expressions floating around in the literature which calculate the local acceleration due to gravity at the center of mass of the vertical column of air above the observer (g_m). The expression for g_m that I have adopted in this work comes from the defintion adopted by the World Geodetic System 1984 (WGS84), with an additional height correction:

$$\begin{aligned} g_m^{WGS84} &= 9.7803267714 \left(\frac{1 + 0.00193185138639 \sin^2(\phi)}{\sqrt{1 - 0.00669437999013 \sin^2(\phi)}} \right) - 0.003086\ H_c\ m/s^2 \\ &= 9.7803267714 \left(\frac{1 + 0.00193185138639 \sin^2(\phi)}{\sqrt{1 - 0.00669437999013 \sin^2(\phi)}} \right) - 0.02253 - \\ &\quad 0.0027774\ h_0\ m/s^2 \end{aligned} \tag{87}$$

where h_0 is the height of the observer and H_c is the height of the center of mass of the vertical column of air above the observer, both in km.

Allen (1964) quotes the following form:

$$\begin{aligned} g_m^{Allen} &= 9.80618 - 0.025865 \cos(2\phi) + 0.000058 \cos^2(2\phi) - 0.00308\ H_c\ m/s^2 \\ &= 9.780373 \left(1 + 0.005289 \sin^2(\phi) - 0.0000059 \sin^2(2\phi) - 0.000315\ H_c\right)\ m/s^2 \\ &= 9.757883 \left(1 + 0.005301 \sin^2(\phi) - 0.0000059 \sin^2(2\phi) - 0.000284\ h_0\right)\ m/s^2 \end{aligned} \tag{88}$$

From the The Explanatory Supplement to the Astronomical Almanac (1992) (which is also the form used in *slalib* and by Hohenkerk & Sinclair (1985)):

$$g_m^{ES} = 9.784\,(1.0 - 0.0026\cos(2\phi) - 0.00028\ h_0)\ \ m/s^2 \tag{89}$$

The Wikipedia "Earth's gravity" entry quotes something called Helmert's equation:

$$\begin{aligned} g_m^{wiki} &= 9.780327\,\big(1 + 0.0053024\sin^2(\phi) - 0.0000058\sin^2(2\phi)\big) - 0.003086\ H_c\ m/s^2 \\ &= 9.757883\,\big(1 + 0.005301\sin^2(\phi) - 0.0000059\sin^2(2\phi) - 0.000284\ h_0\big)\ \ m/s^2 \end{aligned} \tag{90}$$

which appears to be a near exact replication of the Allen (1964) formula. Sinclair (1982) quotes the following:

$$\begin{aligned} g_m^{Sinclair} &= 9.780318\,\big(1 + 0.0053024\sin^2(\phi) - 0.0000058\sin^2(2\phi)\big) - 0.003086\ H_c\ m/s^2 \\ &= 9.757790\,\big(1 + 0.005315\sin^2(\phi) - 0.0000058\sin^2(2\phi) - 0.000284\ h_0\big)\ \ m/s^2 \end{aligned} \tag{91}$$

which is quite close to the Wikipedia version of Helmert's equation and the Allen (1964) equation. The CRC handbook gives yet another variant:

$$\begin{aligned} g_m^{CRC} &= 9.780356\,\big(1 + 0.0052885\sin^2(\phi) - 0.0000059\sin^2(2\phi)\big) - 0.003086\ H_c\ m/s^2 \\ &= 9.757828\,\big(1 + 0.005301\sin^2(\phi) - 0.0000059\sin^2(2\phi) - 0.000284\ h_0\big)\ \ m/s^2 \end{aligned} \tag{92}$$

with the reference Jursa (1985). The web site

`http://geophysics.ou.edu/solid_earth/notes/potential/igf.htm`

lists the following, which is based on the Geodetic Reference System 1967:

$$\begin{aligned} g_m^{IGF67} &= 9.78031846\,\big(1 + 0.0053024\sin^2(\phi) - 0.0000058\sin^2(2\phi)\big) - 0.003086\ H_c\ m/s^2 \\ &= 9.757791\,\big(1 + 0.005315\sin^2(\phi) - 0.0000058\sin^2(2\phi) - 0.000284\ h_0\big)\ \ m/s^2 \end{aligned} \tag{93}$$

where I have added the free-air and height correction term. Finally, Saastamoinen (1972) derives:

$$\begin{aligned} g_m^{Saast} &= 9.8062\,(1 - 0.00265\cos(2\phi) - 0.00031\ H_c) \\ &= 9.784\,(1 - 0.00266\cos(2\phi) - 0.00028\ h_0) \end{aligned} \tag{94}$$

Pat Wallace points out that in fact there is so much variation in g_m from spot to spot on the surface that any of the above formulae are good enough, especially for the refraction application. Intercomparing the equations for g_m above over all latitudes and heights from 0 to 10 km, differences between g_m derived from these equations are < 0.001. Therefore, I have adopted the Saastamoinen (1972) formula given its simplicity.

C Relative Humidity and Saturation Vapor Pressure

Note that the relative humidity at the observer (RH_0, in percent) is related to the saturation vapour pressure (e_{sat}, in mb; Buck (1981)) as follows (see Crane (1976))

$$\begin{aligned} e_{sat} &= \left(1.0007 + 3.46 \times 10^{-6} P_0\right) 6.1121 \exp\left[\frac{17.502 T_0}{T_0 + 240.97}\right] && (95) \\ P_{w0} &= e_{sat} RH_0 \left[1 - (1 - RH_0)\frac{e_{sat}}{P_0}\right]^{-1} && (96) \end{aligned}$$

This relationship between e_{sat}, P_{w0}, and RH_0 comes in handy when using expressions for N_0 which involve linear and quadratic expansions in P_0 and P_{w0}. Unfortunately, this complicated form for e_{sat} does not yield itself to closed-form integration.

By assuming that the relative humidity remains constant throughout the troposphere, and equal to its value at the observer ($RH(r) = RH_0$), we can write:

$$\frac{P_w}{P_{w0}} = \frac{e_{sat}(P, T)}{e_{sat}(P_0, T_0)} \quad (97)$$

Tabulated values of e_{sat} versus T indicate that:

$$\frac{e_{sat}(P, T)}{e_{sat}(P_0, T_0)} = \left(\frac{T}{T_0}\right)^{\gamma} \quad (98)$$

which yields:

$$\frac{P_w}{P_{w0}} = \left(\frac{T}{T_0}\right)^{\gamma} \quad (99)$$

As noted by Sinclair (1982) and Hohenkerk & Sinclair (1985), the power index γ is derived by fitting to the tabulated values of P_{sat} versus T given in Smithsonian (1951). This fit produces the following:

$$P_{sat} = \left(\frac{T}{247.1}\right)^{18.36} \quad (100)$$

Comparing this expression to that derived by Buck (1981) (Equation 95) over the range P = 600 − 1200 mb and T = −30 − +20 C indicates agreement to within ±0.2 mb. Therefore, the approximate relation between P_{sat} and T (Equation 99) represents a good approximation over this relevant range of P and T.

References

Allen, C. W. 1964, "Astrophysical Quantities", second edition, Athlone Press

Auer, L. H. & Standish, E. M. 1979, "Astronomical Refraction: Computational Method for All Zenith Angles", Yale University Astronomy Department note

Auer, L. H. & Standish, E. M. 2000, AJ, 119, 2472

Birch, K. P. & Downs M. J. 1993, Metrologia, 30, 155

Brussaard, G. & Watson, P. A. 1995, in *Atmospheric Modelling and Millimetre Wave Propagation*, p. 254

Buck, A. L. 1981, *Journal of Applied Meteorology*, 20, 1527

Chao, C. C. 1974, JPL Technical Report 32-1587, pp. 61-76

Crane, R. K. 1976, in *Methods of Experimental Physics*, volume 12, part B, page 188, equation 2.5.5

Davis, J. L., Herring, T. A., Shapiro, I. I., Rogers, A. E. E., & Elgered, G. 1985, Radio Science, 20, 6, pp. 1593-1607

Garfinkel, B. 1944, AJ, 50, 169

Garfinkel, B. 1967, AJ, 72, 235

Hohenkerk, C. Y. & Sinclair, A. T. 1985, NAO Technical Note 63, *HM Nautical Almanac Office*

Jursa, A.S. (Ed), "Handbook of Geophysics and the Space Environment 4th Edition", Air Force Geophysics Laboratory, 1985, p14-17

Lanyi, G. 1984, TDA Progress Report 42-78, 152

Liebe, H. J. & Hopponen, J. D. 1977, *IEEE Trans. Antennas Propagation*, AP-25, 336, equation 9

Liebe, H. J. 1989, International Journal of Infrared and Millimeter Waves, 10, No. 6, 631

Liebe, H. J., Hufford, G. A., & Cotton, M. G. 1993, "Propagation Modeling of Moist Air and Suspended Water/Ice Particles at Frequencies Below 1000 GHz", AGARD (Advisory Group for Aerospace Research & Development) Conference Proceedings, 542, 3-1 through 3-10

Livengood, T. A., Fast, K. E., Kostiuk, T., Espenak, F., Buhl, D., Goldstein, J. J., Hewagama, T., & Ro, K. H. 1999, PASP, 111, 512

Marini, J. W. 1972, Radio Science, 7, 223-231

Murray, C. A. 1982, "Vectorial Astrometry", §7.3, Adam Hilger Ltd, Bristol (ed)

Niell, A. E. 1996, Journal of Geophysical Research, 101, B2, pp. 3227-3246

Niell, A. E. 2000, Presentation at COST 716 entitled *Towards Operational GPS Meteorology*, contribution title: "Preliminary Evaluation of Atmospheric Mapping Functions Based on Numerical Weather Models"

Rüeger, J. M. 2002, “Refractive Index Formulae for Radio Waves”, in “JS28: Integration of Techniques and Corrections to Achieve Accurate Engineering”, FIG XXII International Congress, Washington, DC, April 19-26, 2002

Saastamoinen, J. 1972, “Atmospheric Correction for the Troposphere and Stratosphere in Radio Imaging of Satellites”, in “The Use of Artificial Satellites for Geodesy”, Geophysics Monogram Series, 15, S. W. Henriksen *et al.*(ed.), pp. 247-251

Sinclair, A. T. 1982, NAO Technical Note 59, Royal Greenwich Observatory

Smith, E. K. & Weintraub, S. 1953, Proc. IRE, 41, 1035

Sovers, O. J., Fanselow, J. L., & Jacobs, C. S. 1998, Reviews of Modern Physics, 70, No. 4, Pages 1393-1454

Smithsonian Meteorological Tables, 1951, Table 94

The Explanatory Supplement to the Astronomical Almanac (1992), §3.28, p. 142

Thompson, A. R., Moran, J. M., & Swensen, G. W., Jr., 2001, “Interferometry and Synthesis in Radio Astronomy”, Second Edition, Wiley Interscience

Yan, H. & Ping, J. 1995, AJ, 110, 934

Yan, H. 1996, AJ, 112, 1312