MEMORANDUM May 19, 2010

TO: Working Group on Quantization Correction for the ALMA Correlator
FROM: Fred Schwab
SUBJECT: Four-Level van Vleck Correction via Table Lookup

Here are a few notes relevant to the discussion at the two recent meetings of our working group. The most pressing concern is the need to speed up the application of the four-level mode van Vleck correction. At the April 14 meeting the consensus opinion was that we should study whether table look-up, among a 2-D table of pre-computed van Vleck correction curves, might be a viable approach. I agreed to have a look at how large a table would be required.

In the four-level mode of the ALMA correlator, the optimal efficiency,[1] which is equal to 88.1154%, is achieved for a quantizer voltage threshold settings $v_{1x} = v_{1y} = 0.995687$. (These are measured in units of v/σ, where σ_x and σ_y are the r.m.s. voltage levels of the input data streams.) The steps in the calculation of the optimal efficiency are illustrated in Figure 1. The expectation values of the corresponding zero-lag autocorrelations ($r_{xx}(1)$ and $r_{yy}(1)$, say) are equal to 3.55222. Figure 2 illustrates the co-dependence of autocorrelation quantizer voltage threshold and autocorrelation function (ACF) zero-lag expectation value.

Figure 3 shows the four-level cross-correlation efficiency as a function of v_{1x} and v_{1y} (as well as of the equivalent ACF zero-lag expectation values). This two-dimensional dependence is a separable function, the geometric mean of the corresponding one-dimensional autocorrelation efficiency curves. It is handy to have expressions for the ACF zero-lag expectation values as functions of v_{1x} and v_{1y}. These are

$$\langle r_{xx}(1)\rangle = 9 - 8\,\mathrm{erf}\left(\frac{v_{1x}}{\sqrt{2}}\right)\,, \quad \text{and} \tag{1a}$$

$$\langle r_{yy}(1)\rangle = 9 - 8\,\mathrm{erf}\left(\frac{v_{1y}}{\sqrt{2}}\right)\,. \tag{1b}$$

The expectation value of the cross-correlation function (CCF) at $\rho = 1$ is given by

$$\langle r_{xy}(1)\rangle = 9 - 2\,\mathrm{erf}\left(\frac{\min(v_{1x}, v_{1y})}{\sqrt{2}}\right) - 6\,\mathrm{erf}\left(\frac{\max(v_{1x}, v_{1y})}{\sqrt{2}}\right)\,. \tag{2}$$

The latter can also be written as

$$\langle r_{xy}(1)\rangle = \frac{3\min\big(\langle r_{xx}(1)\rangle, \langle r_{yy}(1)\rangle\big) + \max\big(\langle r_{xx}(1)\rangle, \langle r_{yy}(1)\rangle\big)}{4}\,. \tag{3}$$

Relation (3) will be useful if the van Vleck correction of the CCFs is via table look-up from a pre-computed table of normalized van Vleck correction curves. (Use of the normalized curves is probably preferred for table look-up, because the domains of definition of the unnormalized curves would differ throughout the table.)

[1]The efficiency of an n-level digital autocorrelator, in the weak-correlation limit $\rho \to 0$ (under the usual assumptions of zero-mean, band-limited Gaussian input noise of standard deviation σ, sampling at the Nyquist rate, equi-spaced quantizer input voltage thresholds, and equi-spaced quantizer output levels) is given by

$$\eta_n(v_1) = \begin{cases} \dfrac{2\left(1+2\sum_{k=2}^{n/2} e^{-\frac{1}{2}(k-1)^2 v_1^2}\right)}{\pi(n-1)^2 - 8\pi\sum_{k=1}^{(n-2)/2} k\,\mathrm{erf}\big(kv_1/\sqrt{2}\big)}\,, & \text{for } n \text{ even} \\[2ex] \dfrac{8\left(\sum_{k=1}^{(n-1)/2} e^{-\frac{1}{2}(2k-1)^2 v_1^2}\right)}{\pi(n-1)^2 - 4\pi\sum_{k=0}^{(n-3)/2} (2k+1)\,\mathrm{erf}\big(kv_1/\sqrt{2}\big)}\,, & \text{for } n \text{ odd}\,, \end{cases}$$

where v_1 is the first positive quantizer input voltage threshold level (measured in units of v/σ). Digital cross-correlation utilizes a pair of digitizers, with corresponding m-level and n-level quantization functions Q_x and Q_y, characterized by first positive input voltage thresholds v_{1x} and v_{1y}, and in this case the efficiency is simply the geometric mean $\sqrt{\eta_m(v_{1x})\eta_n(v_{1y})}$.

As discussed by Jeff Kern at the recent meetings of this working group, the local-oscillator, digitization/resampling, and time- and frequency-multiplexing signal paths into the ALMA correlator do not incorporate automatic level control feedback circuitry to maintain optimal power levels. Instead, variable attenuators located upstream in the signal path are adjusted periodically, after preliminary processing of data samples acquired at the beginning of each subscan (and, possibly, at intervals within the subscan). There are two types of correlator diagnostics which standardly are used to infer the r.m.s. signal input levels relative to the digital correlator's (fixed) voltage quantization steps: either quantization zone population statistics, or zero-lag signal autocorrelations. The ALMA correlator design does not include the accumulators which would be necessary to provide zonal population statistics. The system must rely, instead, on ACF raw correlation zero-lag values—which are provided by the software data acquisition system—in order to infer r.m.s. voltage levels into the correlator. Here we would use the inverse relations corresponding to Equation 1,

$$v_{1x} = \sqrt{2}\,\mathrm{erf}^{-1}\left(\frac{9-\langle r_{xx}(1)\rangle}{8}\right), \quad \text{and} \tag{4a}$$

$$v_{1y} = \sqrt{2}\,\mathrm{erf}^{-1}\left(\frac{9-\langle r_{yy}(1)\rangle}{8}\right), \tag{4b}$$

to infer the r.m.s. input voltage levels, σ_x and σ_y. These inferred values would then be used to adjust the upstream variable attenuators, which are adjustable in one-quarter decibel steps, as I understand it. They would also be used, post- van Vleck correction, to convert from auto- and cross-correlation coefficient to auto- and cross-covariance (i.e., power). (In the event that an old van Vleck curve is re-used, the most recent ACF zero-lag data should probably be used for the scaling to power.)

On our Wiki page, in the section titled "Background Material", there is a link to a data file showing the actual variations of the zero-lag ACF values across all correlator dumps within a selected sub-scan acquired by the ALMA correlator operating in four-level mode. I am not sure of the exact circumstances of the acquisition of these data. (They are from JIRA ticket AIV-989 on "frequent platforming and scalloping in autocorrelation spectra"; the discussion in the JIRA ticket is too disjointed for me to follow.) I believe that both FDM and TDM data are present in this scan. I show histograms of these data in Figure 4. All the data are shown in the histogram at the top of this figure. Todd somewhere remarked, I think, that the TDM data are the small handful which cluster around $v_1 = 2.3$ in the histogram, and that the FDM data cluster near 1 and near 3.5. Evidently the FDM data near 3.5 were acquired with carefully adjusted attenuator settings, and those data near 1 (where most of the samples lie within the innermost quantization levels) may correspond to hot-load observations with no readjustment of attenuation.

The data with zero-lag values near 3.5 are shown in the bottom histogram of Figure 4. The range of this histogram corresponds to r.m.s. power levels within $\pm 1/2\,\mathrm{dB}$ of optimal. 99.64% of the zero-lag values between 3 and 4 lie within this range, and 98.09% fall within $\pm 1/4\,\mathrm{dB}$ of optimal. For this sub-scan, the mean of the zero-lag values within the range $[3, 4]$ is 3.55174.

A representative four-level van Vleck curve is shown in the top portion of Figure 5. The idea, which I presented in GBT Memo. 250, for efficient van Vleck correction of multi-lag data is to represent the ideal curve via a spline approximation to small number of points computed to high accuracy along the curve, as illustrated by the sixty-five dots shown here. The bottom plot in this figure shows the relative error of the spline interpolant, which in this (typical) case is at most 6×10^{-6}.

Figure 6 illustrates the range of variation of van Vleck cross-correlation correction curves if the input voltage levels are maintained such that the zero-lag ACF levels fall within the range $[3, 4]$. At high correlations, the relative differences between the curves are of order several percent.

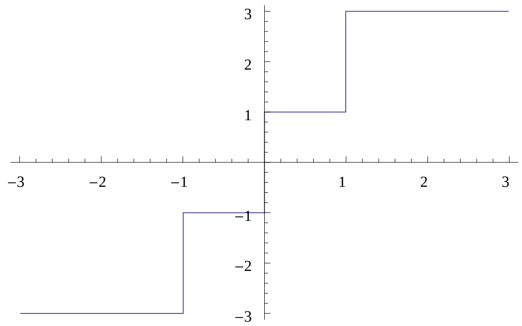
Figure 7 shows the approximation errors that would ensue from a table look-up scheme covering a unit square in ACF zero-lag space, centered at the optimal position, with grid spacing of 1/100. I.e., we are considering here the case of an $N \times N$ grid, for $N = 101$, of pre-computed van Vleck curves. (Owing to symmetry, a table of precomputed van Vleck curves of size $N(N+1)/2$ would suffice.) The typical relative error is seen to be of order $\sim$0.02%, and the maximum relative error is around $\sim$0.08%.

Gianni Comoretto, in ALMA Memo. No. 583, suggests using polynomial approximation (obtained via reversion of the power series for r vs. ρ) for small $|\rho|$, and via the spline approximants only for relatively large ρ. He remarks that "the resulting relation is quite accurate up to $\rho = 0.2$." Figure 8 shows the error of the power series reversion formula (including terms in r, r^3, and r^5) at the corners of the zero-lag space unit square $[3, 4] \times [3, 4]$. Compared with the spline approximants of Figure 7, the polynomial formula evidently would do as well or better for $|\rho| < \sim 0.4$.

This memo has focused on four-level correlator van Vleck correction but has not dealt with the eight-level case. However, I would like to include here two figures pertaining to that case. Figure 9 illustrates the efficiency optimization, and Figure 10 shows a representative eight-level van Vleck correction curve and the relative error curve for a spline interpolant thereof.

In[3]:= **Qplot[4]**

Out[3]=

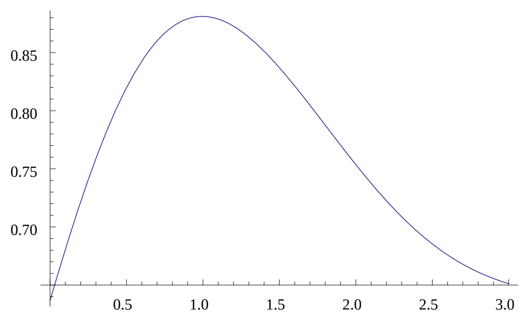


In[4]:= **effevsp[4]**

Out[4]=
$$\frac{2\left(1+2\,e^{-\frac{v1^2}{2}}\right)^2}{\pi\left(9-8\,\mathrm{Erf}\left[\frac{v1}{\sqrt{2}}\right]\right)}$$

In[5]:= **Plot[%, {v1, 0, 3}]**

Out[5]=

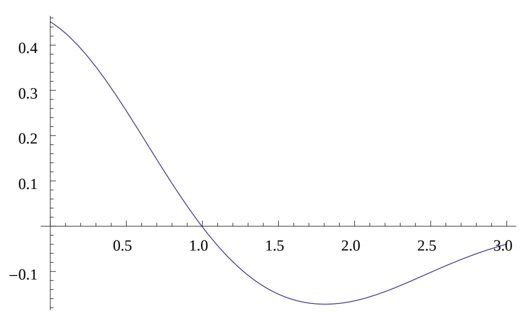


In[6]:= **D[%4, v1]**

Out[6]=
$$\frac{16\sqrt{2}\,e^{-\frac{v1^2}{2}}\left(1+2\,e^{-\frac{v1^2}{2}}\right)^2}{\pi^{3/2}\left(9-8\,\mathrm{Erf}\left[\frac{v1}{\sqrt{2}}\right]\right)^2}-\frac{8\,e^{-\frac{v1^2}{2}}\left(1+2\,e^{-\frac{v1^2}{2}}\right)v1}{\pi\left(9-8\,\mathrm{Erf}\left[\frac{v1}{\sqrt{2}}\right]\right)}$$

In[7]:= **Plot[%, {v1, 0, 3}]**

Out[7]=



In[9]:= **FindRoot[%6, {v1, 1}]**

Out[9]= {v1 → 0.995687}

In[10]:= **%4 /. v1 → 0.995687**

Out[10]= 0.881154

Figure 1. Plots of the four-level correlator quantization function *(top)*, the efficiency curve as function of voltage threshold v_1 *(middle)*, and the derivative of the efficiency with respect to v_1 *(lower plot)* are shown here. Maximum efficiency (88.1154%) occurs at the zero ($v_1 = 0.995687$) of the derivative curve, which is calculated by numerical root-finding.

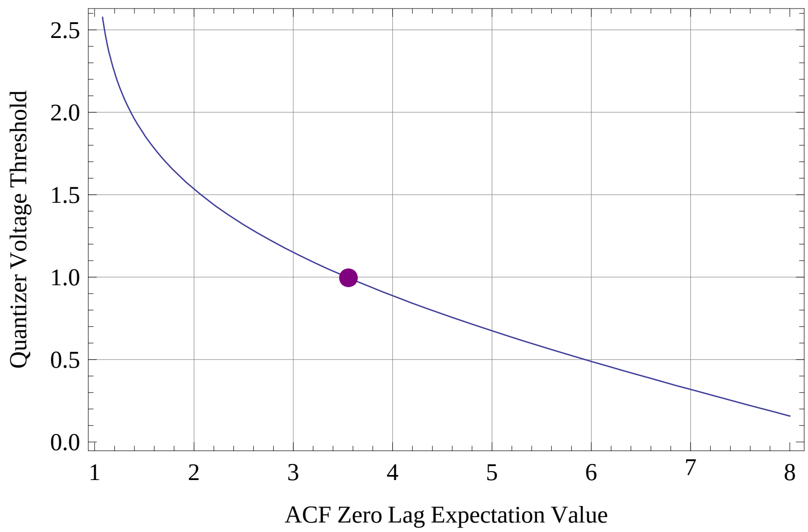
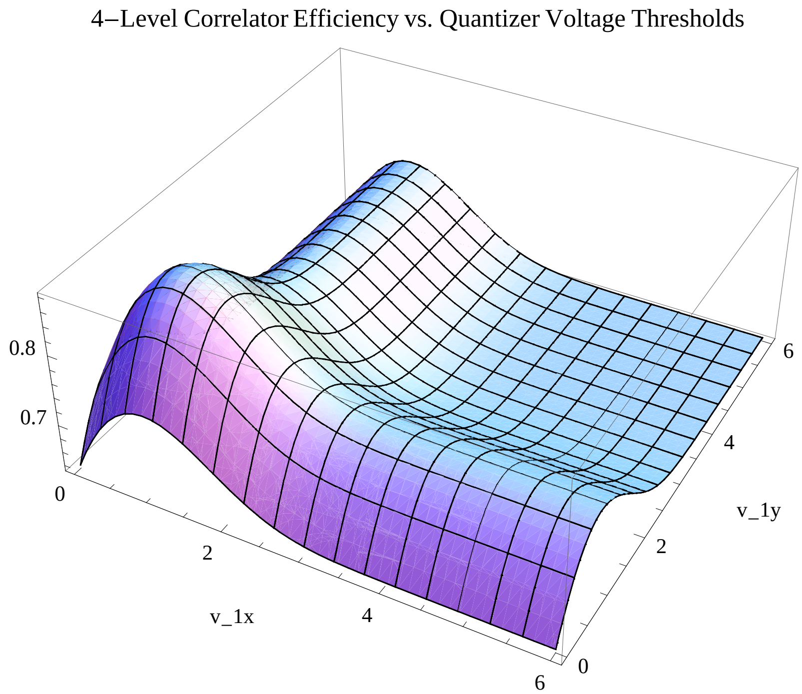


Figure 2. This plot shows the relation between four-level autocorrelator zero-lag expectation value and quantizer voltage threshold parameter, v_1. Here we assume equi-spaced quantizer voltage thresholds $(-v_1, 0, +v_1)$ and equi-spaced output levels $(-3, -1, +1, +3)$. The maximum efficiency, 88.1154%, occurs at $v_1 = v_{\rm opt} = 0.995687$ (shown by the purple dot); the corresponding ACF zero-lag expectation value is 3.55522.



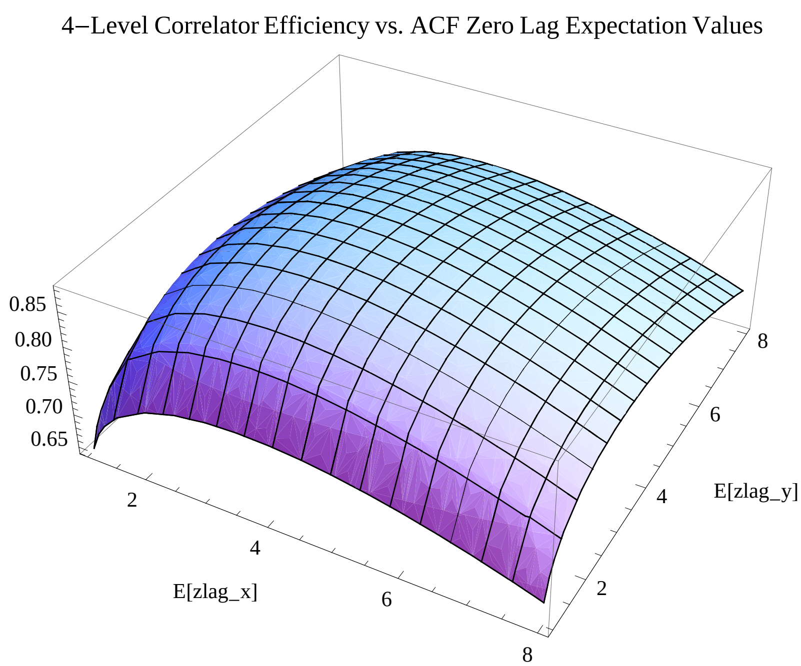


Figure 3. *(Top)* The four-level cross-correlator efficiency is a separable function of quantizer voltage thresholds v_{1x} and v_{1y}, as shown here. *(Bottom)* The relation between four-level correlator efficiency and ACF zero-lag expectation values. The maximum efficiency occurs at $v_{1x} = v_{1y} = v_{\rm opt} = 0.995687$, corresponding to zero-lag expectation values of 3.55522. (*Cf.* Figure 2.)

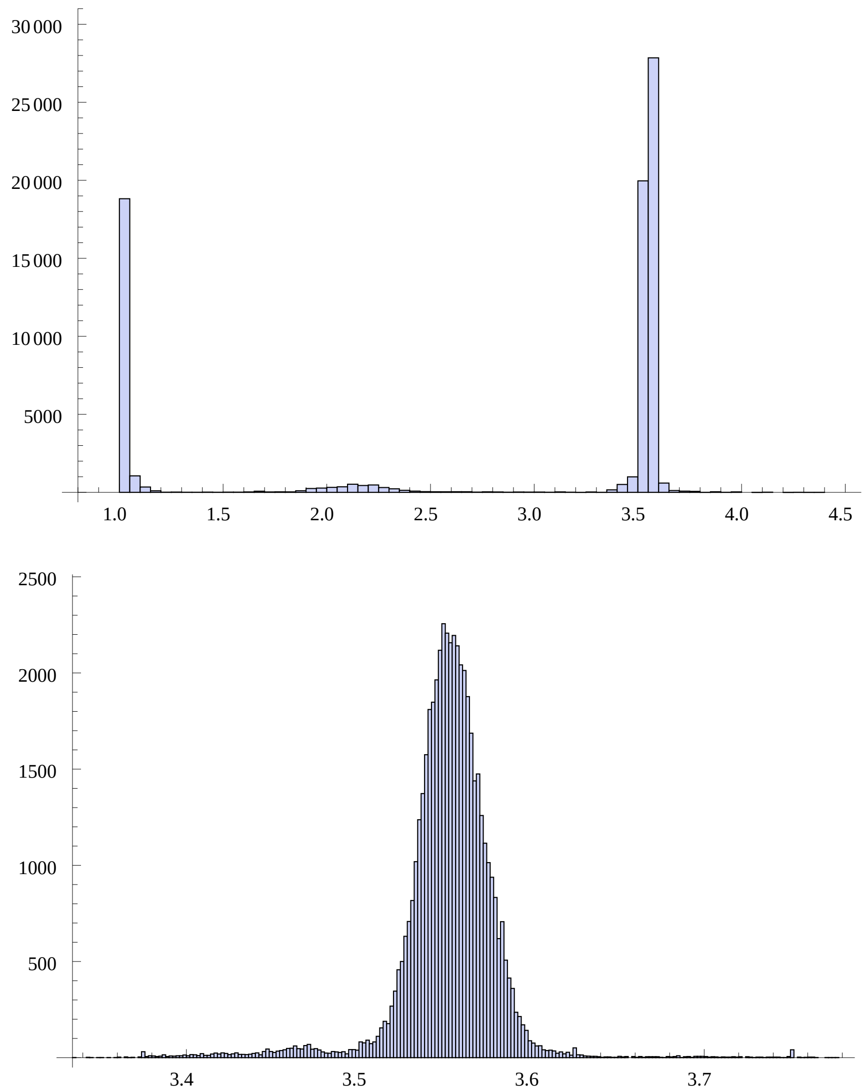
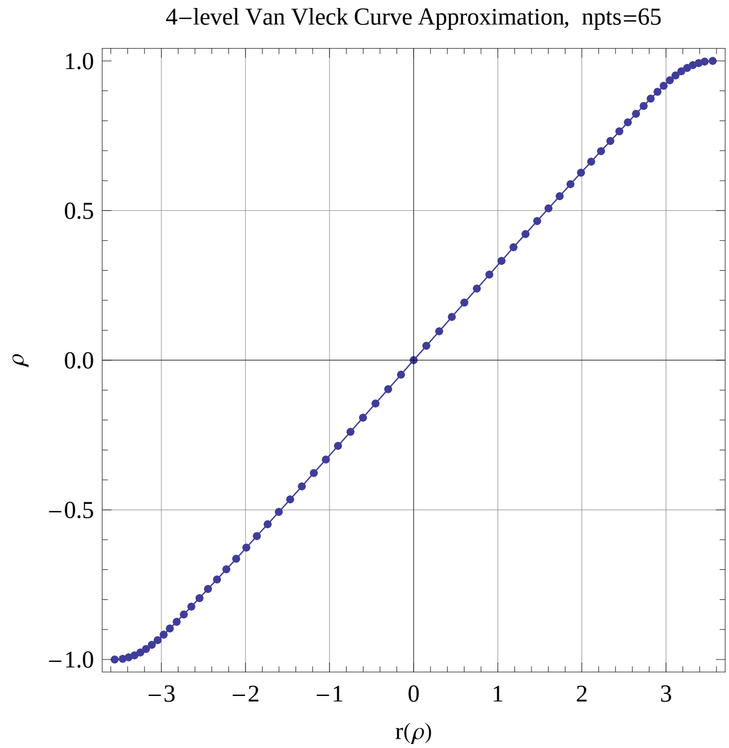


Figure 4. *(Top)* Histogram showing variations of ACF zero-lag values across all correlator dumps in the sub-scan referred to in the section of our Wiki page titled "Background Material". *(Bottom)* Histogram of the cluster near the optimal zero-lag value, 3.55526. The range of this histogram, $[3.33257, 3.44381]$, corresponds to r.m.s. power levels within $\pm 1/2\,\mathrm{dB}$ of optimal; 99.64% of the zero-lag values between 3 and 4 lie within this range; 98.09% fall within $\pm 1/4\,\mathrm{dB}$ of optimal. For this sub-scan, the mean of the zero-lag values within the range $[3, 4]$ is 3.55174.



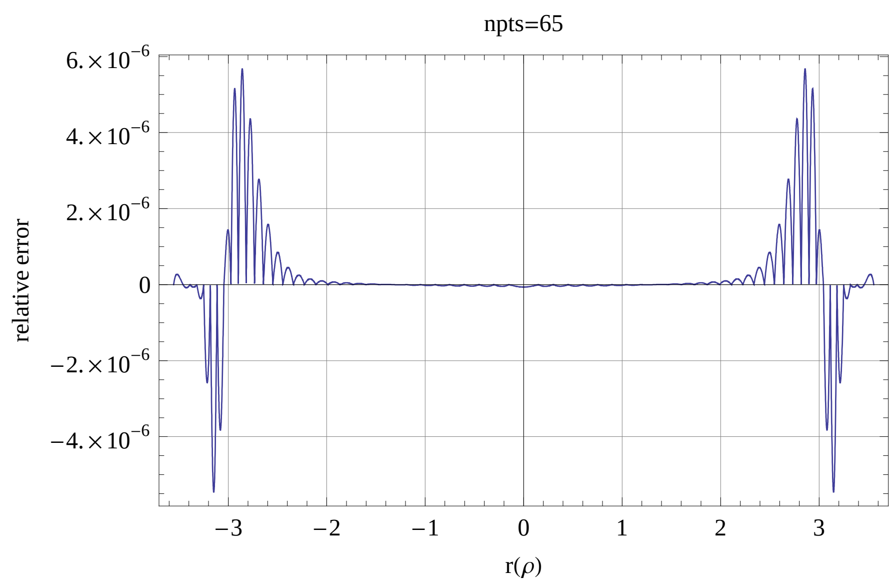
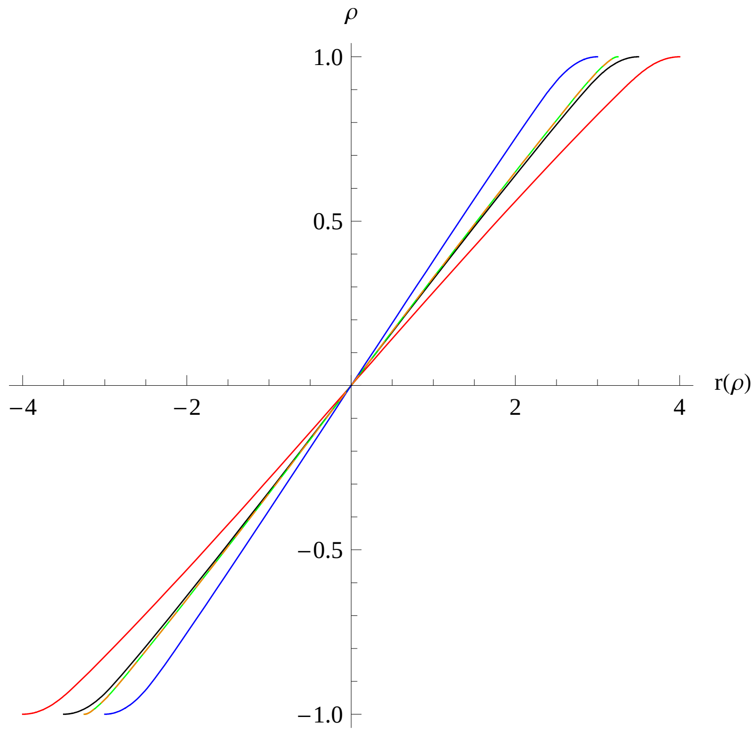


Figure 5. *(Top)* A representative four-level van Vleck correction curve is shown by the solid line. The sixty-five solid dots correspond to shifted Chebyshev sampling points which might be used for spline interpolation. *(Bottom)* The relative error of the cubic natural spline interpolant based on these sixty-five interpolation nodes.



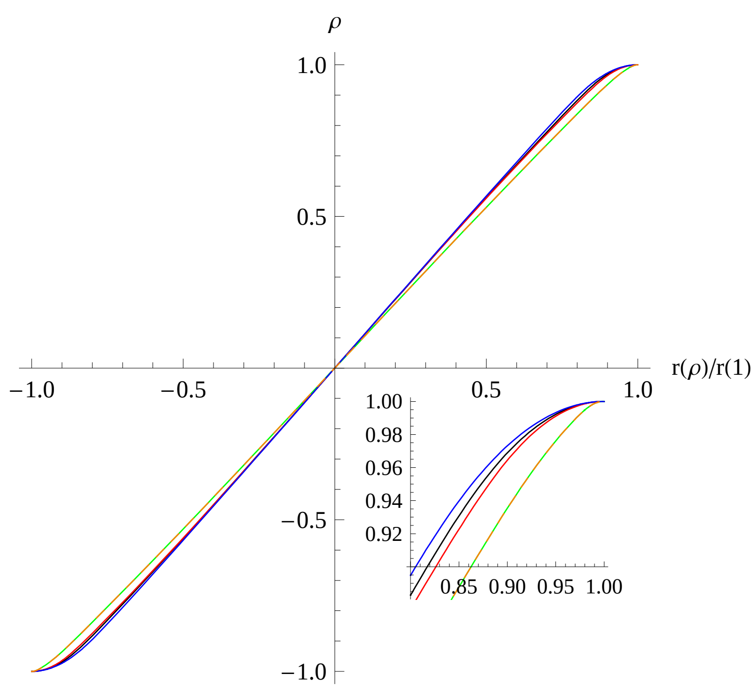


Figure 6. *(Top)* van-Vleck correction curves corresponding to the center *(black)* and the corner extremes *(red, green, blue, and orange, counter-clockwise from upper right corner)* of the unit width tabular grid of Figure 10. *(Bottom)* As above, but showing the normalized van-Vleck curves.

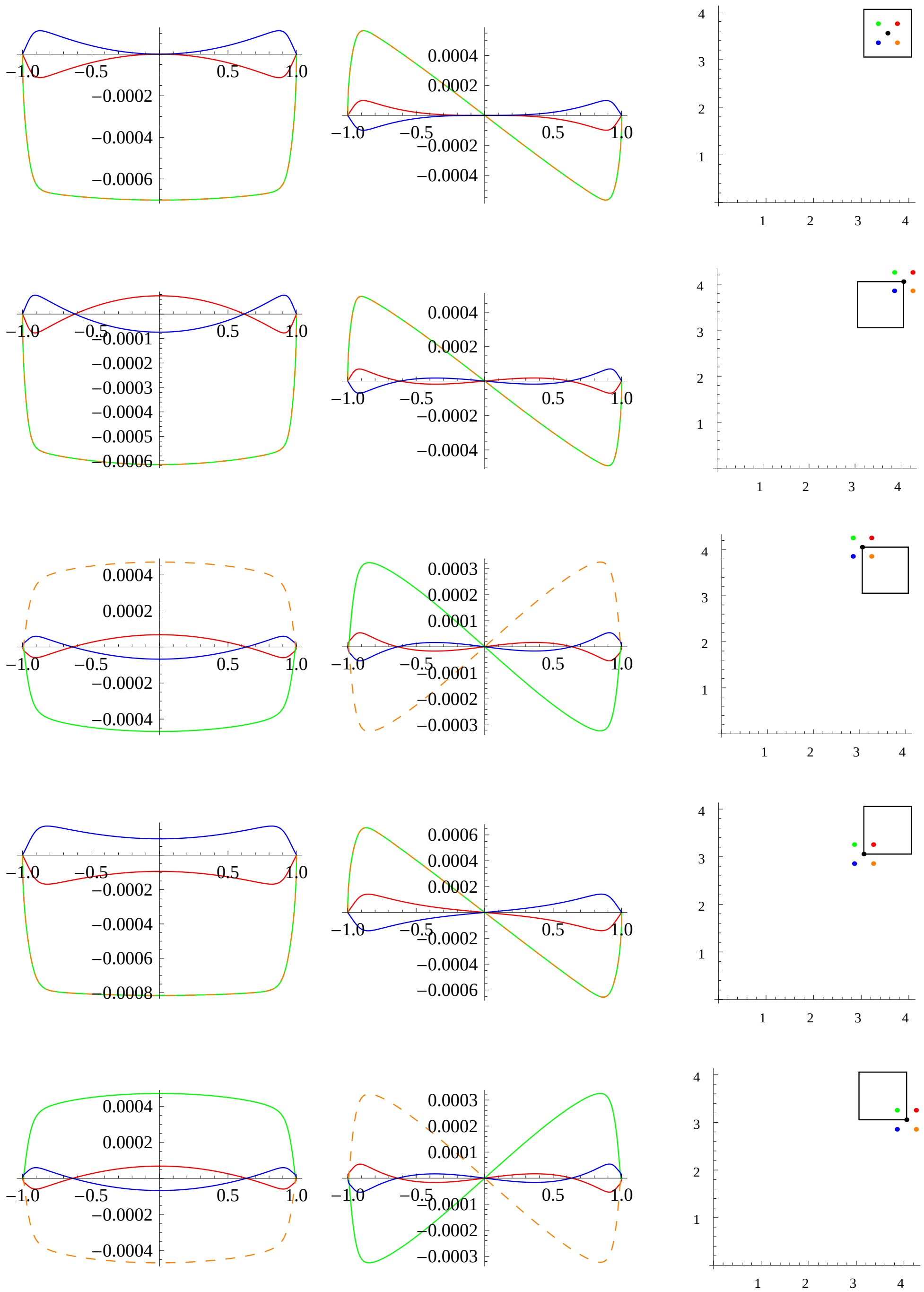


Figure 7. Relative error *(left-hand column)* and absolute error *(middle column)* in 4-level van Vleck correction using nearest-point zero-lag table lookup, assuming a grid spacing $\Delta r(1) = 0.01$. The tabular grid is assumed to cover a square of unit width in ACF zero-lag values *(shown in the right-hand column)*, which is centered at $(3.55522, 3.55522)$. (The number 3.55522 is the expectation of the ACF zero-lag value when the quantizer voltage threshold v_1 (measured in units of v/σ) is set to its optimum value, $v_{\rm opt} \approx 0.995687$.) The error plots in the uppermost row correspond to the four corners of the central grid cell. Those in the other four rows correspond to the corner grid cells—as shown in the right-hand plots, where the direction of the offset is indicated by the colored dots. The same colors are used for the respective error curves. (The orange curve is drawn with dashed lines because it sometimes coincides nearly exactly with the green error curve.) The abscissae in these plots are the true correlation, $\rho(r)$. The maximum relative error over the entire square ought not to exceed that at the four corners, which here is seen to be $\sim$0.08%. (The average relative error is a good deal smaller, perhaps $\sim$0.02%.) For this level of accuracy, then, an $N \times N$ grid of size 101×101 would suffice to cover a numerical range $[3, 4] \times [3, 4]$ of ACF zero-lag values. In fact, owing to symmetry, a table of precomputed van Vleck curves of size $N(N + 1)/2$ would suffice.

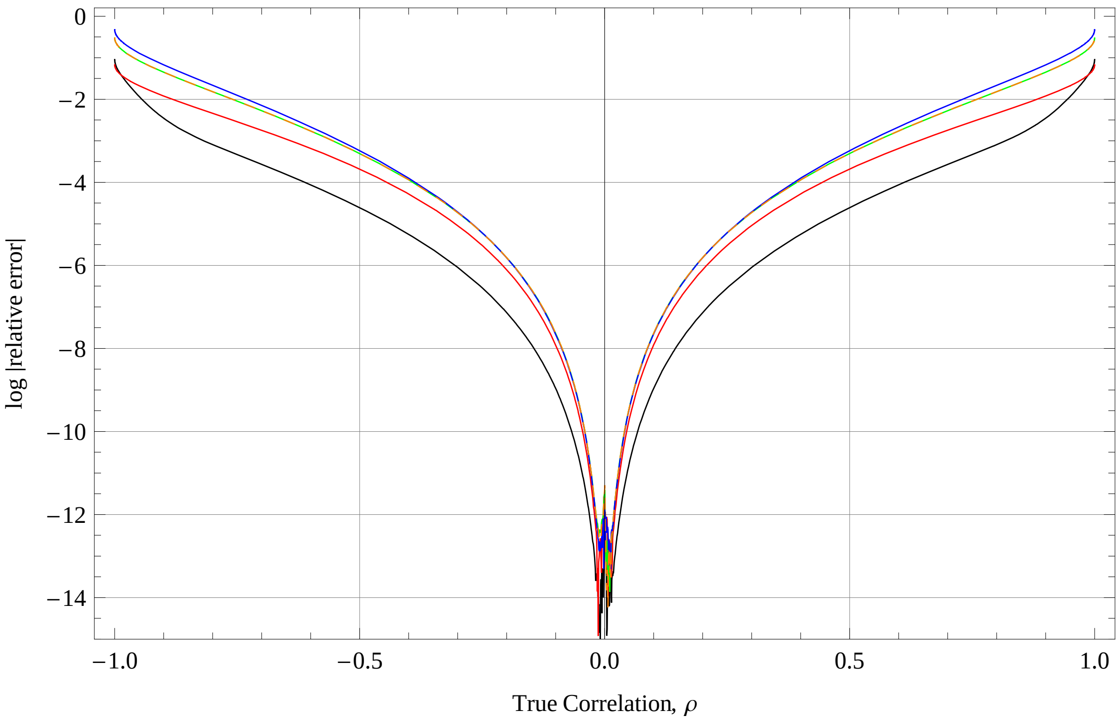
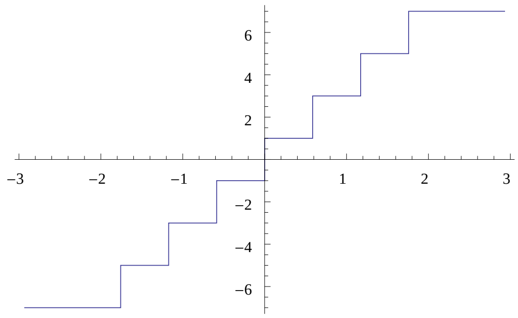


Figure 8. Comoretto (ALMA Memo. No. 583) suggests using polynomial approximation (via reversion of the power series for r vs. ρ) for small $|\rho|$, and via the spline approximation only for large ρ. This figure shows the base ten logarithm of the absolute value of the error of the power series reversion formula, at the corners of the zero-lag space unit square $[3, 4] \times [3, 4]$ (the colored curves) and at the center (black). Compared with the spline approximants of Figure 7, the polynomial formula evidently would do as well or better for $|\rho| < \sim 0.4$.

In[3]:= **Qplot[8]**

Out[3]=



In[4]:= **effevsp[8]**

Out[4]=

$$\frac{2\left(1+2\left(e^{-\frac{9\,v1^2}{2}}+e^{-2\,v1^2}+e^{-\frac{v1^2}{2}}\right)\right)^2}{\pi\left(49-8\left(\mathrm{Erf}\left[\frac{v1}{\sqrt{2}}\right]+3\,\mathrm{Erf}\left[\frac{3\,v1}{\sqrt{2}}\right]+2\,\mathrm{Erf}\left[\sqrt{2}\,v1\right]\right)\right)}$$

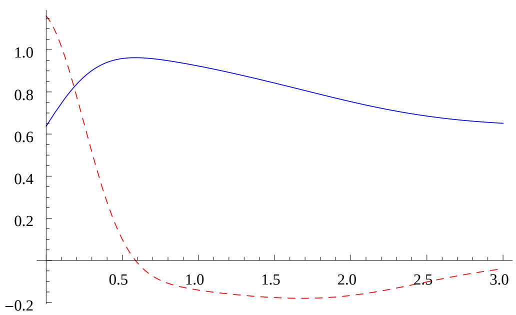
In[5]:= **D[%, v1] // Simplify**

Out[5]=

$$\left(8\,e^{-9\,v1^2}\left(9+4\,e^{\frac{5\,v1^2}{2}}+e^{4\,v1^2}\right)\left(1+2\,e^{-\frac{9\,v1^2}{2}}\left(1+e^{\frac{5\,v1^2}{2}}+e^{4\,v1^2}\right)\right)\right.$$
$$\left.\left(2\left(2+2\,e^{\frac{5\,v1^2}{2}}+2\,e^{4\,v1^2}+e^{\frac{9\,v1^2}{2}}\right)\sqrt{\frac{2}{\pi}}+e^{\frac{9\,v1^2}{2}}\,v1\left(-49+8\,\mathrm{Erf}\left[\frac{v1}{\sqrt{2}}\right]+24\,\mathrm{Erf}\left[\frac{3\,v1}{\sqrt{2}}\right]+16\,\mathrm{Erf}\left[\sqrt{2}\,v1\right]\right)\right)\right)\Big/$$
$$\left(\pi\left(49-8\left(\mathrm{Erf}\left[\frac{v1}{\sqrt{2}}\right]+3\,\mathrm{Erf}\left[\frac{3\,v1}{\sqrt{2}}\right]+2\,\mathrm{Erf}\left[\sqrt{2}\,v1\right]\right)\right)^2\right)$$

In[7]:= **Plot[{%4, %5}, {v1, 0, 3}, PlotStyle → {Blue, {Dashed, Red}}]**

Out[7]=



In[8]:= **FindRoot[%5, {v1, .5}]**

Out[8]= {v1 → 0.586019}

In[9]:= **%4 /. v1 → 0.586019**

Out[9]= 0.96256

Figure 9. Plots of the eight-level correlator quantization function *(upper)*, the efficiency curve as function of voltage threshold v_1 *(lower, blue)*, and the derivative of the efficiency with respect to v_1 *(lower, dashed red)* are shown above. Maximum efficiency (96.256%) occurs at the zero ($v_1 = 0.586019$) of the derivative curve, which is calculated by numerical root-finding.

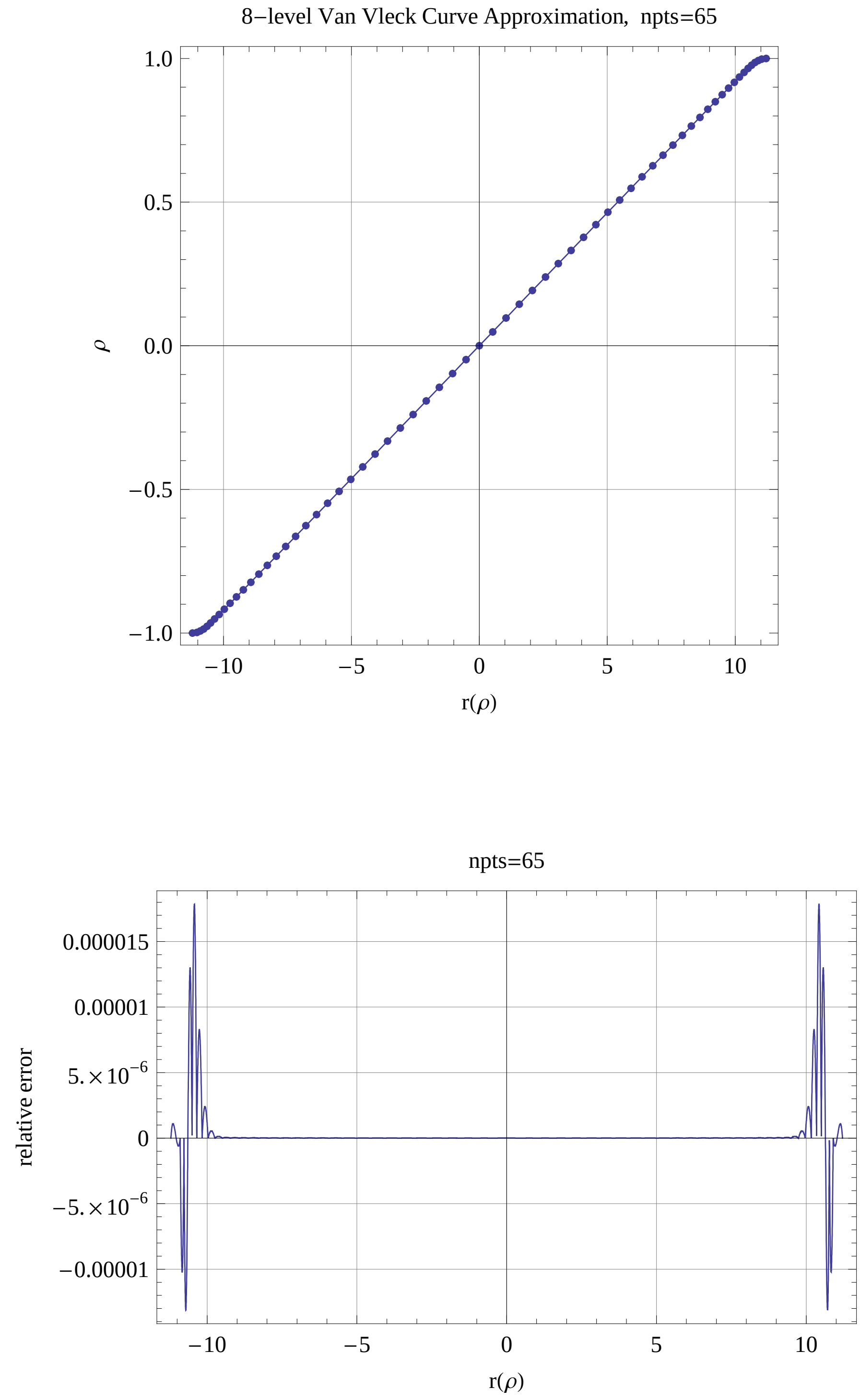


Figure 10. *(Top)* A representative eight-level van Vleck correction curve is shown by the solid line. The sixty-five solid dots correspond to shifted Chebyshev sampling points which might be used for spline interpolation. *(Bottom)* The relative error of the cubic natural spline interpolant based on these sixty-five interpolation nodes.